การคำนวณของตัวแพร่สำหรับตัวแกว่งกวัดฮาร์มอนิกอย่างง่ายคู่ควบกับสนามไฟฟ้าคงที่ผ่านวิธีการของชวิงเงอร์

Calculation of the propagator for a simple harmonic oscillator coupled to a constant electric field via Schwinger's method

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บทคัดย่อ

ในบทความนี้ เราคำนวณตัวแพร่ไฟน์แมนสำหรับตัวแกว่งกวัดฮาร์มอนิกอย่างง่ายควบคู่กับสนามไฟฟ้าคงที่โดยใช้วิธีของชวิงเงอร์ ซึ่งอิงตามผลเฉลยของสมการไฮเซนเบิร์กสำหรับตำแหน่งและตัวดำเนินการโมเมนตัมแบบบัญญัติ ผลเฉลยดังกล่าวจะถูกใช้เพื่อเขียนตัวดำเนินการแฮมิลตันตามอันดับของตัวดำเนินการตำแหน่ง $\hat{X}(\mathrm{O})$ และ $\hat{X}(\mathrm{t})$ การใช้อันดับตัวดำเนินการตามเวลาที่เหมาะสมควบคู่ไปกับเงื่อนไขย่อยและเงื่อนไขเริ่มต้นส่งผลให้ได้ตัวแพร่ดังกล่าว เราพบว่าตัวเผยแพร่ที่ได้รับนั้นสอดคล้องกับตัวแพร่ที่ได้จากการใช้ปริพันธ์ตามวิถีของไฟน์แมนในงานของ Poon และ Muñoz (Poon & Muñoz 1999) เราคาดหวังว่าเทคนิคนี้จะเป็นประโยชน์และเป็นที่ยอมรับอย่างกว้างขวางสำหรับนักศึกษาฟิสิกส์

คำสำคัญ: ปริพันธ์ตามวิถีของไฟน์แมน, ตัวแพร่, วิธีการของชวิงเงอร์, สมการไฮน์เซนเบิร์ก

Abstract

In this article, we compute the Feynman propagator for a simple harmonic oscillator coupled to a constant electric field using Schwinger's method, which is based on the solution of the Heisenberg equations for the position and canonical momentum operators. Such solutions are then used to write the ordered Hamiltonian operator of the position operators $\hat{X}$ (O) and $\hat{X}$ (t). The utilization of proper operator ordering, along with subsidiary and initial conditions, results in the yield of such a propagator. We found that the propagator obtained is consistent with the one obtained using the Feynman path integral in the work of Poon and Muñoz (Poon & Muñoz 1999). We anticipate that this technique will be advantageous and widely recognized for physics students

Keywords: Feynman path integral, propagator, Schwinger's method, Heisenberg equation

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Introduction

The calculation of the propagator for a quantum mechanical system can be approached through various methods. Among these, the most prevalent method involves solving the time-dependent Schrödinger equation. Another technique entails constructing the matrix element of the unitary time operator within the spatial framework. These methodologies, along with others, necessitate a profound knowledge of the Hamiltonian operator. It is fair to say that the Feynman path integral (Feynman, 1948) is a powerful and elegant approach for computing the propagator. This method harnesses the Lagrangian formalism, transforming position and momentum from operators into ordinary classical quantities, such as in the famous textbook by Feynman and Hibbs (Feynman & Hibbs, 1965), where they elucidated the computation of propagators for harmonic oscillators by using the Feynman path integral. Recently Poon and Muñoz (Poon & Muñoz, 1999) employed this technique to compute the non-relativistic propagator for a general quadratic Lagrangian—natural point of departure if one intends to do perturbation theory in the path integral approach. They also applied this approach to calculate the propagator of a simple harmonic oscillator coupled to a constant electric field. A recent research paper by Chaithanapreecha and Yongram (Chaithanapreecha & Yongram, 2023) used the Feynman path integral to calculate the propagator or a damped harmonic oscillator coupled to an electric field. And so on (Cohem, 1998; Brown & Zhang, 1994; Farina, Maneschy & Neves, 1993; Holstein, 1985; Mannheim, 1988).

Moreover, Schwinger (1951) developed a beautiful and powerful method, which is the so-called Schwinger's method (SM), in the context of relativistic quantum field theory to treat effective actions in quantum electrodynamics (QED). However, Schwinger's approach is highly suited for calculating non-relativistic propagators, such as the recent work done by Urrutia and Hernández (1984) using Schwinger's action principle to calculate the Feynman propagator for a damped harmonic oscillator with a time-dependent frequency under a time-dependent external force. To the best of our understanding, subsequent to that time, only a limited number of papers have been authored utilizing this approach, namely: in 1986, Urrutia and Manterola (Urrutia & Manterola, 1986) used it in the problem of an anharmonic charged oscillator under a magnetic field; througout the same calendar year, Horing, Cui, and Fiorenza (Horing, Cui, & Fiorenza, 1986) applied Schwinger's method to obtain the Green function for crossed time-dependent electric and magnetic fields; in 1993, Fararina & Segui-Santonja (1993) published a calculation of the Feynman propagator for a harmonic oscillator with a time-dependent frequency by using Schwinger's method. Rabello & Farina (1995) used a gauge covariant poperatot technique which led to a deduced path integral for a charged particle in an arbitrary stationary magnetic field, verifying the midpoint-rule for the discrete form of the interaction term with the vector potential. For evaluating the small time propagator they used a method developed by Schwinger; Barone, Boschi-Filho & Farina (2003) used Schwinger's method to obtain the Feynman propagator for the nonrelativistic harmonic oscillator; Aragão, Boschi-Filho, Farina, and Barone (Aragão, Boschi-Filho, Farina & Barone, 2007) reconsidered the Feynman propagator of two non-relativistic systems: a charged particle in a uniformed magnetic field and a charged harmonic oscillator in a uniform magnetic field by using Schwinger's method. Instead of solving the Heisenberg equations for the position and the canonical momentum operator, they applied this method by solving the Heisenberg equations for the gauge invariant operators.; Pepore, Kirdmanee, and Sukbot (2017) and Thongpool & Pepore (2022) derived the propagators for a damped harmonic oscillator with time-dependent mass and frequency and a time-dependent inverted harmonic oscillator by using Schwinger's method as well.

As previously stated, Schwinger's approach is commonly employed to derive the propagator of non-relativistic systems. However, it is far less widely utilized compared to the Feynman path integral. To confirm that Schwinger's method is extremely powerful also, our purpose in this paper is to provide the reader with the propagator for a simple harmonic oscillator coupled to a constant electric field that is computed in a straightforward way by Schwinger's method, which is based on the solution of the Heisenberg operator equations of motion. The use of proper operator ordering and the subsidiary and initial conditions yields the propagator for such a system. We then compare the propagator obtained with the one obtained using the Feynman path integral in the work of Poon and Muñoz (1999).

To establish our notation, we write the Feynman propagator for a time independent nonrelativistic system with Hamiltonian operator $\hat{H}$ in the form:

$$K(x_b, x_a; \tau) = \theta(\tau)\langle x_b | \hat{U}(\tau) | x_a \rangle \quad (1)$$

where $\hat{U}(\tau)$ is the time evolution operator:

$$\hat{U}(\tau) = \exp\left(-i\hat{H}\tau / \quad \right) \quad (2)$$

and $\theta(\tau)$ is the step function defined by

$$\theta(\tau) = \begin{cases} 1 & \text{if } \tau \geq 0 \\ 0 & \text{if } \tau < 0 \end{cases} \quad (3)$$

First, observe that for $\tau > 0$, Eq.(1) leads to the differential equation for the Feynman propagator:

$$i \frac{\partial}{\partial \tau} K(x_b, x_a; \tau) = \langle x_b | \hat{H} \exp\left(-\frac{i}{} \hat{H}\tau\right) | x_a \rangle \quad (4)$$

By using the general relation between operators in the Heisenberg and Schrödinger pictures,

$$\hat{O}_H(t) = e^{i\hat{H}t/} \; \hat{O}_S e^{-i\hat{H}t/} \quad (5)$$

it is not difficult to show that if $|x\rangle$ is an eigen-vector of the operator $\hat{X}$ with eigenvalue x, then it is also true that

$$\hat{X}(t)|x,t\rangle = x|x,t\rangle \quad (6)$$

where

$$\hat{X}(t) = e^{i\hat{H}t/} \; \hat{X} e^{-i\hat{H}t/} \quad (7)$$

and $|x,t\rangle$ is defined as

$$|x,t\rangle = e^{i\hat{H}t/} \; |x\rangle \quad (8)$$

Using this notation, the Feynman propagator can be written as:

$$K(x_b, x_a; \tau) = \langle x_b, \tau | x_a, 0 \rangle \quad (9)$$

where

$$\hat{X}(\tau)|x_b, \tau\rangle = x_b |x_b, \tau\rangle \quad (10a)$$

$$\hat{X}(0)|x_a, 0\rangle = x_a |x_a, 0\rangle \quad (10b)$$

The differential equation for the Feynman propagator, Eq.(4), takes the form

$$i \frac{\partial}{\partial \tau} \langle x_b, \tau | x_a, 0 \rangle = \langle x_b, \tau | \hat{H} | x_a, 0 \rangle \quad (\tau > 0) \quad (11)$$

The form of Eq. (11) is very suggestive and is the starting point for the very elegant operator method introduced by Schwinger. The main idea is to calculate the matrix element on the right-hand side of Eq. (11) by writing $\hat{H}$ in terms of the operators $\hat{X}(\tau)$ and $\hat{X}(0)$, appropriately ordered. Schwinger's method can be summarized by the following steps:

(i) Solve the Heisenberg equations for the operators $\hat{X}(\tau)$ and $\hat{P}(\tau)$, which are given by:

$$i\hbar \frac{\partial}{\partial t} \hat{X}(t) = [\hat{X}(t), \hat{H}], i\hbar \frac{\partial}{\partial t} \hat{P}(t) = [\hat{P}(t), \hat{H}] \quad (12)$$

Equations (12) follow directly from Eq. (5).

(ii) Use the solutions obtained in step (1) to rewrite the Hamiltonian operator $\hat{H}$ as a function of the operators $\hat{X}(0)$ and $\hat{X}(\tau)$ ordered in such a way that in each term of $\hat{H}$, the operator $\hat{X}(\tau)$ must appear on the left-hand side, while the operator $\hat{X}(0)$ must appear on the right-hand side. This ordering can be done easily with the help of the commutator $[\hat{X}(0),\ \hat{X}(\tau)]$ (see Eq. (25)). We shall refer to the Hamiltonian operator written in this way as the ordered Hamiltonian operator $\hat{H}_{ord}(\hat{X}(\tau),\ \hat{X}(0))$. After this ordering, the matrix element on the right-hand side of Eq. (11) can be readily evaluated:

$$\begin{aligned}\langle x_b,\tau|\hat{H}|x_a,0\rangle &= \langle x_b,\tau|\hat{H}_{ord}\big(\hat{X}(\tau),\hat{X}(0)\big)|x_a,0\rangle \\ &\equiv H(x_b,x_a;\tau)\langle x_b,\tau|x_a,0\rangle\end{aligned} \quad (13)$$

where we have defined the function H. The latter is a c-number and not an operator. If we substitute this result in Eq. (11) and integrate over τ, we obtain:

$$\begin{aligned}\langle x_b,\tau|x_a,0\rangle = C(x_b,x_a)\exp\Big(-\frac{i}{\hbar}\int^{\tau} d\tau' \\ \times H(x_b,x_a;\tau')\Big)\end{aligned} \quad (14)$$

where $C(x_b,x_a)$ is an arbitrary integration constant.

(iii)The last step is devoted to the calculation of $C(x_b,x_a)$. Its dependence on x_b and x_a can be determined by imposing the following conditions:

$$\langle x_b,\tau|\hat{P}(\tau)|x_a,0\rangle = -i\ \frac{\partial}{\partial x_b}\langle x_b,\tau|x_a,0\rangle \quad (15a)$$

$$\langle x_b,\tau|\hat{P}(0)|x_a,0\rangle = +i\ \frac{\partial}{\partial x_a}\langle x_b,\tau|x_a,0\rangle \quad (15b)$$

These equations come from the definitions in Eq. (10) together with the assumption that the usual commutation relations hold at any time:

$$[\hat{X}(\tau),\hat{P}(\tau)] = [\hat{X}(0),\hat{P}(0)] = i \quad (16)$$

After using Eq. (15), there is still a multiplicative factor to be determined in $C(x_b,x_a)$. This can be done simply by imposing the propagator initial condition:

$$\lim_{\tau\to 0^+}\langle x_b,\tau|x_a,0\rangle = \delta(x_b - x_a) \quad (17)$$

Derivation of the propagator

We start the calculation of the propagator for a simple harmonic oscillator coupled to a constant electric field by using the Schwinger method. The Hamiltonian operator of this system can be written as

$$\hat{H} = \frac{\hat{p}^2(\tau)}{2m} + \frac{1}{2}m\omega^2\hat{X}^2(\tau) - qE\hat{X}(\tau) \quad (18)$$

where m is the mass of the particle, ω is natural frequency of oscillation, q is an electric charged, and E is an electric field. We rewrite Eq. (18) as the time-independent Hamiltonian operator, it reads

$$\hat{H} = \frac{\hat{p}^2(0)}{2m} + \frac{1}{2}m\omega^2\hat{X}^2(0) - qE\hat{X}(0) \quad (19)$$

despite the fact that the operator and are explicitly time dependent. It is matter of choice whether to work with the Hamiltonian operator given by Eq.(18) or by Eq.(19). For simplicity, we choose the latter.

As stated in step (i), we start by writing down the corresponding Heisenberg equations:

$$\frac{d}{dt}\hat{X}(t) = \frac{\hat{P}(t)}{m} \quad (20a)$$

$$\frac{d}{dt}\hat{P}(t) = -m\omega^2\hat{X}(t) + qE \quad (20b)$$

whose solutions permit us to write for $t = \tau$ that

$$\hat{X}(\tau) = \hat{X}(0)\cos\omega\tau + \frac{\hat{P}(0)}{m\omega}\sin\omega\tau - \frac{2qE}{m\omega^2}\sin^2\frac{\omega\tau}{2} \quad (21)$$

For later convenience, we also write the corresponding expression for $\hat{P}(\tau)$:

$$\begin{aligned}\hat{P}(\tau) = -m\omega\hat{X}(0)\sin\omega\tau + \hat{P}(0)\cos\omega\tau \\ + \frac{qE}{\omega}\sin\omega\tau\end{aligned} \quad (22)$$

To complete step (ii) we need to rewrite $\hat{P}(0)$ in terms of $\hat{X}(\tau)$ and $\hat{X}(0)$, which can be done directly from Eq.(21):

$$\hat{P}(0)=\frac{m\omega}{\sin\omega\tau}\left[\hat{X}(\tau)-\hat{X}(0)\cos\omega\tau-\frac{2qE}{m\omega^2}\sin^2\frac{\omega\tau}{2}\right] \quad (23)$$

If we substitute this result into Eq. (19), we obtain

$$\begin{aligned}\hat{H} = &\frac{m\omega^2}{2\sin^2\omega\tau}\Big[\hat{X}^2(\tau)+\hat{X}^2(0)\cos^2\omega\tau \\ &-\hat{X}(0)\hat{X}(\tau)\cos\omega\tau-\hat{X}(\tau)\hat{X}(0)\cos\omega\tau \\ &+\frac{4qE\hat{X}(0)}{m\omega^2}\cos\omega\tau\sin^2\frac{\omega\tau}{2} \\ &-\frac{4qE\hat{X}(\tau)}{m\omega^2}\sin^2\frac{\omega\tau}{2}+\frac{4q^2E^2}{m^2\omega^4}\sin^4(\omega\tau/2)\Big] \\ &+\frac{1}{2}m\omega^2\hat{X}^2(0)-qE\hat{X}(0)\end{aligned} \quad (24)$$

Note that the third term in Eq. (24) is not written in the appropriate order. By using the commutation relation

$$\begin{aligned}\left[\hat{X}(0),\hat{X}(\tau)\right]=&\Big[\hat{X}(0),\hat{X}(0)\cos\omega\tau+\frac{\hat{P}(0)}{m\omega}\sin\omega\tau \\ &-\frac{2qE}{m\omega^2}\sin^2\left(\omega\tau/2\right)\Big] \\ =&\frac{i}{m\omega}\sin\omega\tau\end{aligned} \quad (25)$$

It follows immediately that

$$\hat{X}(0)\hat{X}(\tau)=\hat{X}(\tau)\hat{X}(0)+\frac{i}{m\omega}\sin\omega\tau \quad (26)$$

If we substitute Eq. (26) into Eq. (24), we obtain the ordered Hamiltonian:

$$\begin{aligned}\hat{H}_{ord}=&\frac{m\omega^2}{2\sin^2\omega\tau}\Big[\hat{X}^2(\tau)+\hat{X}^2(0)-2\hat{X}(\tau)\hat{X}(0)\cos\omega\tau \\ &+\frac{4q^2E^2}{m^2\omega^4}\sin^4\frac{\omega\tau}{2}-\frac{4qE\hat{X}(0)}{m\omega^2}\sin^2\frac{\omega\tau}{2} \\ &-\frac{4qE\hat{X}(\tau)}{m\omega^2}\sin^2\left(\omega\tau/2\right)\Big]-\frac{i\ \omega}{2}\cot\omega\tau\end{aligned} \quad (27)$$

Once the Hamiltonian operator is appropriately ordered, we can find the function $H(x_b,x_a;\tau)$ directly from its definition, given by Eq. (13):

$$\begin{aligned}H(x_b,x_a;\tau)=&\frac{\langle x_b,\tau|\hat{H}|x_a.0\rangle}{\langle x_b,\tau|x_a.0\rangle} \\ =&\frac{m\omega^2}{2}\Big[\left(x_b^2+x_a^2\right)\csc^2\omega\tau-2x_bx_a\cot\omega\tau\csc\omega\tau \\ &+\frac{4q^2E^2}{m^2\omega^4}\sin^4\frac{\omega\tau}{2}\csc^2\omega\tau-\frac{4qEx_a}{m\omega^2}\sin^2\frac{\omega\tau}{2}\csc^2\omega\tau \\ &-\frac{4qEx_b}{m\omega^2}\sin^2\frac{\omega\tau}{2}\csc^2\omega\tau\Big]-\frac{i\ \omega}{2}\cot\omega\tau\end{aligned} \quad (28)$$

By using Eq. (14), we can express the propagator in the following form:

$$\begin{aligned}\langle x_b,\tau|x_a,0\rangle=&C(x_b,x_a)\exp\Big\{-\frac{i}{}\int^{\tau}d\tau'\Big[\frac{m\omega^2}{2} \\ &\times\left(\left(x_b^2+x_a^2\right)\csc^2\omega\tau'-2x_bx_a\cot\omega\tau'\csc\omega\tau'\right) \\ &+\frac{4q^2E^2}{m^2\omega^4}\sin^4\frac{\omega\tau'}{2}\csc^2\omega\tau'-\frac{4qE(x_a+x_b)}{m\omega^2} \\ &\times\sin^2\frac{\omega\tau'}{2}\csc^2\omega\tau'-\frac{i\ \omega}{2}\cot\omega\tau'\Big]\Big\}\end{aligned} \quad (29)$$

The integration over τ' in Eq. (29) can be readily evaluated:

$$\begin{aligned}\langle x_b,\tau|x_a,0\rangle=&\frac{C(x_b,x_a)}{\sqrt{\sin\omega\tau}}\exp\Big\{\frac{im\omega}{2\ \sin\omega\tau}\Big(\left(x_b^2+x_a^2\right) \\ &\times\cos\omega\tau 2-x_bx_a\Big)+\frac{iqE}{2\ \omega\sin\omega\tau} \\ &\times\Big[4\Big(x_b+x_a-\frac{qE}{m\omega^2}\Big)\sin^2\frac{\omega\tau}{2} \\ &+\frac{qE\tau}{m\omega}\sin\omega\tau\Big]\Big\}\end{aligned} \quad (30)$$

where $C(x_b,x_a)$ is an arbitrary integration constant to be determined according to step (iii).

The determination of $C(x_b,x_a)$ is done with the aid of Eqs. (15) and (17). However, we need to rewrite the operators $\hat{P}(0)$ and $\hat{P}(\tau)$ in terms of the operators $\hat{X}(\tau)$ and $\hat{X}(0)$, appropriately ordered. For $\hat{P}(0)$ this task has already been done (see Eq.(23)), and for $\hat{P}(\tau)$ we find after substituting Eq. (23) into Eq. (22):

$$\begin{aligned}\hat{P}(\tau)=&m\omega\cot\omega\tau\left[\hat{X}(\tau)-\hat{X}(0)\cos\omega\tau\right] \\ &-m\omega\hat{X}(0)\sin\omega\tau+\frac{qE}{\omega}\sin\omega\tau\end{aligned} \quad (31)$$

Then, by inserting Eqs. (31) and (30) into Eq. (15a) it is not difficult to show that:

$$\frac{\partial C(x_b, x_a)}{\partial x_b} = 0 \tag{32}$$

Analogously, by substituting Eqs.(23) and (30) into Eq. (15b) we have that $\partial C(x_b,x_a)/\partial x_a = 0$. The last two relations tell us that $C(x_b,x_a) = C$, that is, it is a constant independent of x_b and x_a. In order to determine the value of C, we first take the limit $\tau \to 0^+$ on $\langle x_b,\tau \,|\, x_a,0\rangle$. If we use Eq.(30), we find that

$$\begin{aligned}\lim_{\tau\to 0^+}\langle x_b,\tau|x_a,0\rangle &= \lim_{\tau\to 0^+}\frac{C}{\sqrt{\omega\tau}}\exp\left[\frac{im}{2\tau}(x_b - x_a)^2\right] \\ &= C\sqrt{\frac{2\pi i}{m\omega}}\,\delta(x_b - x_a)\end{aligned} \tag{33}$$

If we compare this result with the initial condition, Eq. (17), we obtain $C = \sqrt{m\omega / 2\pi i}$. By substituting this result for C into Eq.(30), we obtain the desired Feynman propagator for the harmonic oscillator:

$$\begin{aligned}&K(x_b, x_a;\tau) = \langle x_b,\tau|x_a,0\rangle \\ &= \sqrt{\frac{m\omega}{2\pi i\hbar\sin\omega\tau}}\exp\Big\{\frac{im\omega}{2\hbar\sin\omega\tau}\Big[(x_b^2 + x_a^2)\cos\omega\tau \\ &-2x_bx_a\Big] + \frac{iqE}{2\,\omega\sin\omega\tau}\Big[4\Big(x_b+x_a - \frac{qE}{m\omega^2}\Big)\sin^2\frac{\omega\tau}{2} \\ &+\frac{qE\tau}{m\omega}\sin\omega\tau\Big]\Big\}\end{aligned} \tag{34}$$

Conclusions

We found the Feynman propagator of a simple harmonic oscillator connected to a constant electric field using Schwinger's method, which is usually used in quantum field theory but also works well for non-relativistic quantum mechanical problems, even though they don't happen very often. Schwinger's method is based on the solution of the Heisenberg equations for the position and canonical momentum operators. Such solutions are then used to write the ordered Hamiltonian operator of the position operators $\hat{X}$ (O) and $\hat{X}$ (t). The utilization of proper operator ordering, along with subsidiary and initial conditions, results in the yield of such a propagator. We found that the propagator obtained is consistent with the one obtained using the Feynman path integral in the work of Poon and Muñoz (1999). Schwinger's method is known for its strong focus on operator formalism and its applications in quantum field theory as well as non-relativistic quantum theory, whereas the path integral method is renowned for its probabilistic interpretation and its broad applicability to various quantum systems, making it an extremely powerful tool in both theoretical and computational physics. We hope that this pedagogical paper may be useful for undergraduate as well as graduate students and that a simple harmonic oscillator coupled to a constant electric field may enlarge the small list of non-relativistic problems that have been treated by such a powerful and elegant method.

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