การค�านวณของตัวแพร่ส�าหรับตัวแกว่งกวัดฮาร์มอนิกอย่างง่ายคู่ควบกับสนามไฟฟ้ าคงที่ ผาน่ วิธีการของชวิงเงอร

Calculation of the propagator for a simple harmonic oscillator coupled to a constant electric field via Schwinger's method

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Received: 21 August 2023 ; **Revised**: 19 October 2023 ; **Accepted**: 3 November 2023

บทคัดย่อ

ในบทความนี้ เราคำนวณตัวแพร่ไฟน์แมนสำหรับตัวแกว่งกวัดฮาร์มอนิกอย่างง่ายควบคู่กับสนามไฟฟ้าคงที่โดยใช้วิธีของชวิงเงอร์ ้ซึ่งอิงตามผลเฉลยของสมการไฮเซนเบิร์กสำหรับตำแหน่งและตัวดำเนินการโมเมนตั้มแบบบัญญัติ ผลเฉลยดังกล่าวจะถูกใช้ เพื่อเขียนตัวดำเนินการแฮมิลตันตามอันดับของตัวดำเนินการตำแหน่ง \hat{X} (O) และ \hat{X} (t) การใช้อันดับตัวดำเนินการตามเวลา ที่เหมาะสมควบคู่ไปกับเงื่อนไขย่อยและเงื่อนไขเริ่มต้นส่งผลให้ได้ตัวแพร่ดังกล่าว เราพบว่าตัวเผยแพร่ที่ได้รับนั้นสอดคล้อง ที่เหมาะสมควบคู่ไปกับเงื่อนไขย่อยและเงื่อนไขเริ่มต้นส่งผลให้ได้ตัวแพร่ดังกล่าว เราพบว่าตัวเผยแพร่ที่ได้รับนั้นสอดคล้อง
กับตัวแเพร่ที่ได้จากการใช้ปริพันธ์ตามวิถีของไฟน์แมนในงานของ Poon และ Muñoz (Poon & Muñoz 1999) เราคาดห เทคนิคนี้จะเป็นประโยชน์และเป็นที่ยอมรับอย่างกว้างขวางสำหรับนักศึกษาฟิสิกส์

ี **คำสำคัญ**: ปริพันธ์ตามวิถีของไฟน์แมน, ตัวแพร่, วิธีการของชวิงเงอร์, สมการไฮน์เซนเบิร์ก

Abstract

In this article, we compute the Feynman propagator for a simple harmonic oscillator coupled to a constant electric field using Schwinger's method, which is based on the solution of the Heisenberg equations for the position and canonical momentum operators. Such solutions are then used to write the ordered Hamiltonian operator of the position operators \hat{X} (O) and \hat{X} (t). The utilization of proper operator ordering, along with subsidiary and initial conditions, results in the yield of such a propagator. We found that the propagator obtained is consistent with the one obtained using the Feynman path integral in the work of Poon and Muñoz (Poon & Muñoz 1999). We anticipate that this technique will be advantageous and widely recognized for physics students

Keywords: Feynman path integral, propagator, Schwinger's method, Heisenberg equation

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Introduction

The calculation of the propagator for a quantum mechanical system can be approached through various methods. Among these, the most prevalent method involves solving the time-dependent Schrödinger equation. Another technique entails constructing the matrix element of the unitary time operator within the spatial framework. These methodologies, along with others, necessitate a profound knowledge of the Hamiltonian operator. It is fair to say that the Feynman path integral (Feynman, 1948) is a powerful and elegant approach for computing the propagator. This method harnesses the Lagrangian formalism, transforming position and momentum from operators into ordinary classical quantities, such as in the famous textbook by Feynman and Hibbs (Feynman & Hibbs, 1965), where they elucidated the computation of propagators for harmonic oscillators by using the Feynman path integral. Recently Poon and Muñoz (Poon & Muñoz, 1999) employed this technique to compute the non-relativistic propagator for a general quadratic Lagrangian—natural point of departure if one intends to do perturbation theory in the path integral approach. They also applied this approach to calculate the propagator of a simple harmonic oscillator coupled to a constant electric field. A recent research paper by Chaithanapreecha and Yongram (Chaithanapreecha & Yongram, 2023) used the Feynman path integral to calculate the propagator or a damped harmonic oscillator coupled to an electric field. And so on (Cohem, 1998; Brown & Zhang, 1994; Farina, Maneschy & Neves, 1993; Holstein, 1985; Mannheim, 1988).

Moreover, Schwinger (1951) developed a beautiful and powerful method, which is the so-called Schwinger's method (SM), in the context of relativistic quantum field theory to treat effective actions in quantum electrodynamics (QED). However, Schwinger's approach is highly suited for calculating non-relativistic propagators, such as the recent work done by Urrutia and Hernández (1984) using Schwinger's action principle to calculate the Feynman propagator for a damped harmonic oscillator with a time-dependent frequency under a time-dependent external force. To the best of our understanding, subsequent to that time, only a limited number of papers have been authored utilizing this approach, namely: in 1986, Urrutia and Manterola (Urrutia & Manterola, 1986)

used it in the problem of an anharmonic charged oscillator under a magnetic field; througout the same calendar year, Horing, Cui, and Fiorenza (Horing, Cui, & Fiorenza, 1986) applied Schwinger's method to obtain the Green function for crossed time-dependent electric and magnetic fields; in 1993, Fararina & Segui-Santonja (1993) published a calculation of the Feynman propagator for a harmonic oscillator with a time-dependent frequency by using Schwinger's method. Rabello & Farina (1995) used a gauge covariant poperatot technique which led to a deduced path integral for a charged particle in an arbitrary stationary magnetic field, verifying the midpoint-rule for the discrete form of the interaction term with the vector potential. For evaluating the small time propagator they used a method developed by Schwinger; Barone, Boschi-Filho & Farina (2003) used Schwinger's method to obtain the Feynman propagator for the nonrelativistic harmonic oscillator; Aragão, Boschi-Filho, Farina, and Barone (Aragão, Boschi-Filho, Farina & Barone, 2007) reconsidered the Feynman propagator of two non-relativistic systems: a charged particle in a uniformed magnetic field and a charged harmonic oscillator in a uniform magnetic field by using Schwinger's method. Instead of solving the Heisenberg equations for the position and the canonical momentum operator, they applied this method by solving the Heisenberg equations for the gauge invariant operators.; Pepore, Kirdmanee, and Sukbot (2017) and Thongpool & Pepore (2022) derived the propagators for a damped harmonic oscillator with time-dependent mass and frequency and a time-dependent inverted harmonic oscillator by using Schwinger's method as well.

is commonly employed to derive the propagator of non-relativistic systems. However, it is far less widely utilized compared to the Feynman path integral. To
confirm that Osburings is acthor in subservable assumist schill that cominger's method is externery perform. with the propagator for a simple harmonic oscillator coupled to a constant electric field that is computed in a straightforward way by Schwinger's method, which is based on the solution of the Heisenberg operator equations of motion. The use of proper operator ordering
equations of motion. The use of proper operator ordering and the subsidiary and initial conditions yields the confirm that Schwinger's method is extremely powerful As previously stated, Schwinger's approach

 $T_{\rm eff}$ differential equation for the Feynman propagator, $T_{\rm eff}$

propagator for such a system. We then compare the propagator obtained with the one obtained using the Feynman path integral in the work of Poon and Muñoz (1999). $\mathsf{1999}.$ (1999) . (1999) . **the** $\frac{1}{2}$ $\frac{1}{$ (1999) . (1999) . propagator for such a system. We then compa propagator obtained with the one obtained usicommunity paint integration and work of 1 contrained in the work of 1 contrained and **x** t^{*x*} t^{*x*} t^{*x*} as defined as α opagator for such a system. We then compare the pagant in the same and grown the armore compare and frequency eynman path integral in the work of Poon and Muñoz 99 . 99). 999). oscillator with time-dependent mass and frequency using Schwinger
Schwinger

 (1127) .
To establish our notation, we write the Feynman propagator for a time independent nonrelativistic system with Hamiltonian operator \hat{H} in the form: computed in a straightforward way by Schwinger's written as:
written as:

$$
K(x_b, x_a; \tau) = \theta(\tau) \langle x_b | \hat{U}(\tau) | x_a \rangle
$$
 (1)
\n
$$
K(x_b, x_a; \tau) = \langle x_b, \tau | x_a, 0 \rangle
$$

\nwhere $\hat{U}(\tau)$ is the time evolution operator:

where $U(\tau)$ is the time evolution operator θ $\hat{\mathbf{r}}$ where $\hat{U}(\tau)$ is the time evolution operator: $\sum_{i=1}^n a_i = a_i$ for a constant experience. confirm that $\hat{r}(\cdot)$ is extremely method is extremely defined in \hat{r} where $U(\tau)$ is the time evolution operator: the reader with the propagator for a simple harmonic for a simple harmonic \mathcal{L}_max

$$
\hat{U}(\tau) = \exp\left(-i\hat{H}\tau / \quad \right) \tag{2}
$$
\n
$$
\hat{X}(\tau)|x_b, \tau\rangle = x_b |x_b, \tau\rangle
$$
\n
$$
\hat{Y}(\tau)|x_b, \tau\rangle = x_b |x_b, \tau\rangle
$$

and θ (τ) is the step function defined by $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and θ (T) is the step function defined by $\frac{d}{dx}$ $\frac{d}{dx}$ and $\theta(\tau)$ is the step function defined by Feynman propagator for a time independent

$$
\theta(\tau) = \begin{cases} 1 & \text{if } \tau \ge 0 \\ 0 & \text{if } \tau < 0 \end{cases}
$$
 (3)

 (4) and (5) and (6) (7) (9) , \Box By using the general relation between operators in the general relation of the general property in the contract of the general state of th Trist, observe that for $t > 0$, Eq.(1) leads
differential equation for the Feynman propagator: First, observe that for $\tau \negthinspace > \negthinspace 0,$ Eq.(1) leads to the $\frac{1}{2}$ differential equation for the Feynman propagator: First, observe that for $V > 0$, Eq.(1) leads to the differential equation for the Feynman propagator: ˆ (, ;) () () *Kx x x U x b a b a* = (1) nonrelativistic system with Hamiltonian operator *H*ˆ That, about the the $\frac{1}{2}$ $\frac{1}{2}$, $\frac{1}{2}$, $\frac{1}{2}$ and the feynman propagator: Feynman propagator for a time independent integral in the work of Poon and Muñoz (1999). rst, observe that for $\tau \geq 0,$ Eq.(1) leads to the ilerential equation for the Feynman propagator:

$$
i \frac{\partial}{\partial \tau} K(x_b, x_a; \tau) = \langle x_b | \hat{H} \exp\left(-\frac{i}{\tau} \hat{H} \tau\right) | x_a \rangle
$$
 (4)

By using the general relation between operators in the Heisenberg and Schrödinger pictures, ina bombangor plote modifiger pictures, By using the general relation between the Heisenberg and Schrödinger pictures,

$$
\hat{O}_H(t) = e^{i\hat{H}t/\hat{O}_S}e^{-i\hat{H}t/\hat{O}_S}
$$
\n(5)

it is not difficult to show that if $|x\rangle$ is an eigenvector of the operator \hat{X} with eigenvalue x , then it is also true that frue that \overline{a} with eigenvalue \overline{a} with eigenvalue \overline{a} and \overline{a} and it is also in its also in it vector of the operator \hat{X} with eigenvalue X , then it is also
true that $\frac{d}{dt}$ and $\frac{d}{dt}$ it is not difficult to show that if $|x\rangle$ is an eigen- θ that for the Feynman propagator: θ ctor of the operator \cancel{X} with eigenvalue \mathcal{X} , then it is al:
ıe that $\frac{1}{2}$ it is not difficult to snow that if $\frac{1}{2}$ is an eigen-

$$
\hat{X}(t)|x,t\rangle = x|x,t\rangle \tag{6}
$$

where $\frac{1}{\sqrt{2}}$ iHttps:// where *<i>xt xt xt <i>xt <i>xt* $$ where $\frac{1}{2}$ Where \overline{a} **• e**re *ⁱ i Kx x x H H x* where $$ where $\overline{}$

$$
\hat{X}(t) = e^{i\hat{H}t} \hat{X} e^{-i\hat{H}t}
$$
 (7)

The
$$
|x, t\rangle
$$
 is defined as

$$
|\infty z|
$$

\n
$$
|x,t\rangle = e^{i\hat{H}t/2} |x\rangle
$$
 (8)

nan
em Using this notation, the Feynman propagator can is the starting point for the very elegant operator (11) ˆ , ,0 , ,0 (0) *ba b a i x x x Hx* ⁼ ˆ , ,0 , ,0 (0) *ba b a i x x x Hx* m cosing this hotation, the regnman propagator can
be written as: α written as: *X X Xt Xt xt xxxxxxii, and it cynthall propagale.*
X 28' ° ∪Sing this hotation, the Feynman propagator
he written as:

$$
K(x_b, x_a; \tau) = \langle x_b, \tau | x_a, 0 \rangle \tag{9}
$$

X () and ˆ $\frac{1}{2}$ s method can be summarized by the summarized by t of Eq. (11) by writing \mathbf{H}^* in the operators of the operators o where. The main is set to be main interested by Schwinger. The main is set of the main interested in the main interest method introduced by Schwinger. The main is seen in the main interaction in the main interaction in the main is \mathcal{L} where $\frac{1}{\sqrt{2}}$ where $\frac{1}{\sqrt{2}}$ where $\frac{1}{\sqrt{2}}$ $\frac{1}{\sqrt$ where *Kx x x x* (, ;) , ,0 *ba b a* = (9) where $x + y = 0$ written as: *Kx x x x* (, ;) , ,0 *ba b a* = (9)

written as:

(2)
$$
\hat{X}(\tau)|x_b, \tau\rangle = x_b|x_b, \tau\rangle
$$
 (10a)

$$
\hat{X}(0) | x_a, 0 \rangle = x_a | x_a, 0 \rangle \tag{10b}
$$

 ˆ ˆ ˆ ˆ ˆˆ *i Xt Xt H i Pt Pt H* () (), , () (), the differential equation for the Feynm
3) (3) propagator, Eq.(4), takes the form ˆ ˆ ˆ ˆ ˆˆ *i Xt Xt H i Pt Pt H* () (), , () (), following steps: The differential constitution for the Four referred. Squarent is the population of the and *i* yields opagator, Eq. (4), takes the form $\frac{1}{2}$ the starting point for the very element operator $\frac{1}{2}$ The differential equation for the Feynr The differential equation for the Feynman

propagator Fg.(4), takes the form $(1,1)$ propagator, Eq.(4), takes the form
propagator, Eq.(4), takes the form pagator, Eq.(4), takes the form

$$
\text{the} \qquad i \quad \frac{\partial}{\partial \tau} \langle x_b, \tau | x_a, 0 \rangle = \langle x_b, \tau | \hat{H} | x_a, 0 \rangle \quad (\tau > 0) \tag{11}
$$

֦ *The form of Eq. (11) is verv suggestive and* ⁷/² (11) is very suggestive and
the starting point for the very elegant operator met $\overline{}$ introduced by Schwinger. The main idea is to **u** the matrix element on the right-hand side of **F**

∧ () The form of Eq. (11) is ver
(4) the statung point for the very elegant operator me
introduced by Schwinger. The main idea is to calc Introduced by Scribinger. The main idea is to calculate
ors the matrix element on the right-hand side of Eq. (11) by writing \hat{H} in terms of the operators $\hat{X}(\tau)$ and $\hat{X}(0)$ appropriately ordered. Schwinger's method can be
summarized by the following stons: (12) *t t* (5) summarized by the following steps: \mathcal{A} the starting point for the very elegant operator method operators ˆ *^X* () and ˆ The form of Eq. (11) is very suggestive and is It starting point for the very elegant operator method introduced by Schwinger. The main idea is to calculate in terms of the operators of th the matrix element on the right-hand side of Eq. (11) by
writing \hat{H} in terms of the operators $\hat{X}(\tau)$ and $\hat{X}(0)$,

J (12) (1) GONC dic Tickethology Γ (i) the solutions obtained in step (1) to the solution of Γ (i) Solve the Heisenberg equations for the (i) Solve the Heisenberg equations for the operators $\hat{X}(\tau)$ and $\hat{P}(\tau)$, which are given by:

^{so}
$$
i\hbar \frac{\partial}{\partial t} \hat{X}(t) = [\hat{X}(t), \hat{H}], i\hbar \frac{\partial}{\partial t} \hat{P}(t) = [\hat{P}(t), \hat{H}](12)
$$

way that in each term of *H*ˆ , the operator ^ˆ *X* () rewrite the Hamiltonian operator *H*ˆ as a function of Equations (12) follow directly from Eq. (3). $\mathcal{L}(\mathbf{1}, \mathbf{1}, \mathbf{1}) = \mathcal{L}(\mathbf{1}, \mathbf{1}, \mathbf$ $Equations (12)$ follow directly from Equations (12). Equations (12) follow directly from Eq. (5). *t* the *t t t t*

I *X* () ordered in such a rewrite the Hamiltonian operator \hat{H} as a function of the rewrite the Hamiltonian operator *H*ˆ as a function of (ii) Use the solutions obtained in step (1) to Fewrite the Hamiltonian operator *H* as a function of the negative the Hamiltonian operator *H* as a function of the negative $\hat{Y}(0)$ and $\hat{Y}(\tau)$ ordered in such a way the operators $X(0)$ and $X(0)$ ordered in such a way that
in each term of \hat{H} , the operator $\hat{X}(\tau)$ must appear on the left-hand side, while the operator $\hat{X}(0)$ must appear (ii) Use the solutions obtained in step (1) to \hat{v} on the right-hand side. This ordering can be done easily operators $\hat{X}(0)$ and $\hat{X}(\tau)$ ordered in such a way that with the help of the commutator $\left[\hat{X}(0),\ \hat{X}(\tau)\right]$ (see Eq. (25)). We shall refer to the Hamiltonian operator written in this way as the ordered Hamiltonian operator
 $\hat{H} = (\hat{\hat{Y}}(\tau), \hat{Y}(0))$ After this ordering, the matrix $\hat{H}_{ord}(\hat{X}(\tau),\ \hat{X}(0))$. After this ordering, the matrix ${}_{\mathsf{fi}}$ element on the right-hand side of Eq. (11) can be readily
explicited: evaluated: readily evaluated: element on the right-hand side of Eq. (11) can be readily of
evaluated. readily evaluated: $\mathcal{H}(\mathcal{A})$ and **Exercise field.** We rewrite Equation field. We rewrite written in this way as the ordered Hamiltonian operator si
 $\hat{H} = (\hat{V}(\tau), \hat{V}(0))$. After this ordering, the matrix () ˆˆ ˆ (), (0) *HX X ord* . After this ordering, the matrix shall refer to the Hamiltonian operator written in this control operator written in this c commutator **v** $\frac{\partial \text{ord}(X,Y)}{\partial Y}$ are the right-hand side of Eq. (11) can be readily *I* (0) must appear on the right-hand side. This is a side of the right-hand side. This is a side of the right-hand side. The right-hand side of the right-hand side of the right-hand side of the right-hand side of the ri *X* (0) must appear on the right-hand side. The right-hand side of \mathbb{R} and side. This is seen that \mathbb{R} is seen that \mathbb{R}

$$
\langle x_b, \tau | \hat{H} | x_a, 0 \rangle = \langle x_b, \tau | \hat{H}_{ord} \big(\hat{X}(\tau), \hat{X}(0) \big) | x_a, 0 \rangle
$$

\n
$$
\hat{H} = \frac{\hat{p}^2(\tau)}{2m} + \frac{1}{2m}
$$

\n
$$
= H(x_b, x_a; \tau) \langle x_b, \tau | x_a, 0 \rangle
$$

\n
$$
\hat{H} = \frac{\hat{p}^2(\tau)}{2m} + \frac{1}{2m}
$$

\nwhere *m* is the

where we have defined the function H . The latter is a c-number and not an operator. If we substitute this result in Eq. (11) and integrate over τ , we obtain: $\overline{\mathbf{r}}$ *x Hx x H X X x* and the contract of where we have defined the function H . The \qquad na ter is a c-number and not an operator. If we substitute ε s result in Eq. (11) and integrate over τ , we obtain: \overline{r} where we have defined the function H . The s result in Eq. (11) and integrate over τ , we obtain: where we have defined the function H . The

$$
\langle x_b, \tau | x_a, 0 \rangle = C(x_b, x_a) \exp\left(-\frac{i}{\tau} \int^{\tau} d\tau'\n\times H(x_b, x_a; \tau')\right)
$$
\n(14)
$$
\hat{H} = \frac{\hat{p}^2(0)}{2m}
$$

where $C(x_{\cdot},x_{\cdot})$ is an arbitrary integra constant. where $C(x_b, x_a)$ is an arbitrary integration constant $\frac{d}{dx}$ constant. $\frac{1}{2}$ integration where $C(x_b, x_a)$ is an arbitrary integration

patent α c-number and not an operator. If we substitute this we substitute this we substitute this we substitute this we substitute that α where $C(x_b, x_a)$ is an arbitrary integration. where $C(x_{\cdot} . x_{\cdot})$ is an arbitrary integration a co-number and not and not an operator. If we substitute this we substitute this we substitute this substitut where $C(x_{_b\!x_{_d})$ is an arbitrary integrati where we have defined the function \mathcal{H} . The latter is the latter in \mathcal{H} . The latter is the latter is the latter where $C(x, x)$ is an arbitrary inte $\int_{B}^{B} \frac{d^{2}}{d^{2}}$ where $C(x_b, x_a)$ is an arbitrary integration *b a b ord a*

 $C(x_i, x_i)$. Its dependence on x_i and x_i can be determined to the calculation by imposing the following conditions: ا
| iii)The last step is devoted to the calculation of $C(x_{i}, x_{a})$. Its dependence on x_{i} and x_{a} can be determined $(x_{i}x_{a})$. Its dependence on x_{i} and x_{a} can be determined F
imposing the following conditions: \sim (14) (x, y, x_a) is an architecty integration.

Standard (iii)The last step is devoted to the calculation of
 (x, x) Its dependence on x and x can be determined *x i x x i x* imposing the following conditions: (iii) the last step is devoted to the calculation
x ..x). Its dependence on x, and x can be determined. w_h^{new} . The dependence on h_h^{max} and h_d^{max} can be determined \mathbf{r} c-number and not an operator. If we substitute this we substitute this we substitute this we substitute that \mathbf{r}

the corresponding Heisenberg (15a)
\n
$$
\langle x_b, \tau | \hat{P}(\tau) | x_a, 0 \rangle = -i \frac{\partial}{\partial x_b} \langle x_b, \tau | x_a, 0 \rangle
$$
\n(15a)
\n
$$
\frac{d}{dt} \hat{X}(t) = \frac{\hat{P}(t)}{m}
$$

$$
\langle x_b, \tau | \hat{P}(0) | x_a, 0 \rangle = +i \frac{\partial}{\partial x_a} \langle x_b, \tau | x_a, 0 \rangle
$$
\n(15b)\n
$$
\frac{d}{dt} \hat{X}(t) = \frac{P(t)}{m}
$$
\n
$$
\frac{d}{dt} \hat{P}(t) = -ma
$$

These equations come from the definitions in \overline{F} (10) together with the assumption ¹ ¹ ¹ ² ² *commutation relations hold at any time:* Eq. (10) together with the assumption that the usual
commutation relations hold at any time: commutation relations hold at any time: ˆ , (0) ,0 , ,0 *b a b a a x* s come from the definitions in These equations come from the de
x (10) teaching with the assumetime that the contract of hese equations come from the definitions me from the definitions in
issumption that the usual $\frac{1}{2}$ on relations hold at any time: (15) These equations come from the definitions: in mmutation relations hold at any time:

(iii)The last step is devoted to the calculation relations to the calculation relation relations. These equations come from the definitions in 1. (10) together with the assumption that the usual

$$
\left[\hat{X}(\tau), \hat{P}(\tau)\right] = \left[\hat{X}(0), \hat{P}(0)\right] = i \qquad (16)
$$
\n
$$
\hat{X}(\tau) = \hat{X}(0)\cos\omega\tau + \frac{\hat{P}(0)\cos\omega\tau}{ma}
$$
\nFor later,

After using Eq. (15), there is still a multipl
factor to be determined in $C(x, x)$. This can be After using Eq. (15), there is still a multiplicative α condition: First using Eq. (15), there is sun a manipheative corresponding expression for $P(T)$
factor to be determined in $C(x_p, x_q)$. This can be done condition: simply by imposing the propagator initial condition: After using Eq. (10), there is suit a multiplicative
tor to be determined in $C(x_{\nu}, x_{\nu})$. This can be done to the set determined in $C(x_b, x_a)$. This can be done
the usual the usua commutation relations hold at any time $\frac{1}{2}$ *a* $\frac{3}{x}$ is the determined in $C(x, x)$. This can α **i** α *x*_{*b*}, x_a). This can be done
 α *x*_b imposing the propagator initial condition:

$$
\lim_{\tau \to 0^+} \langle x_b, \tau | x_a, 0 \rangle = \delta(x_b - x_a) \tag{17}
$$

$\frac{1}{\sqrt{2\pi}}$ ϵ Derivation of the propagator **Derivation of the propagator Derivation of the propagator**

m operator
an operator we start the calculation of the propagator for a
an operator simple harmonic oscillator coupled to a constant electric P We start the calculation of the proparticular coupled to a const

F **A** *F m n* field by using the Schwinger method. The Hamiltonian
operator of this system can be written as mere by using the committee method. The ri frequency of any system can be whiten as ² can be readily operator of this system can be written as where **many is the matrix** of the particle of We start the calculation of the propagator for a
simple harmonic oscillator coupled to a constant electric 22 V electric field by using the Schwinger method. The a simple harmonic oscillator coupled to a constant field by using the Schwinger method. The Hamiltonian We start the calculation of the propagator for a we start the calculation of the propagator for a
harmonic oscillator coupled to a constant electric **Derivation of the properties** \mathcal{L}_{max}

$$
\hat{H} = \frac{\hat{p}^{2}(\tau)}{2m} + \frac{1}{2}m\omega^{2}\hat{X}^{2}(\tau) - qE\hat{X}(\tau)
$$
 (18)
(13)

where m is the mass of ded in the fact that the conduct that the a is an electric $\frac{1}{2}$ is an electric charged,
If we substitute and E is an electric field. We rewrite Eq. (18) as the $\frac{du}{dx}$ by an electric field. We few the Eq. (18) or by time-independent Hamiltonian operator. It reads $\langle 0 \rangle$ (13)
where *m* is the mass of the particle, ω is nction H . The chatural frequency of oscillation, q is an electric character of choice whether to work the work of choice when the work of choice when the t, we optairi. The latter independent Hamiltonian operator, it $\frac{1}{2}$ is an electric field. We few
time independent Hamiltonian operator it roads ume-muependent. Frammonian operator, it reads where *m* is the mass of the part
atural frequency of oscillation, *q* is an electri where *m* is the mass of the particle, ω is
natural frequency of oscillation, *q* is an electric charged,
and *F* is an electric field. We rewrite Eq. (18) as the *m* time-independent Hamiltonian operator, it reads
 $\frac{2200}{1}$ where *m* is the mass of the particle, ω

patural fractionian can be set of oscillation *d* is an electric charge and E is an electric field. We rewrite Eq. (18) as the particle mass of the particle and E ime-independent Hamiltonian operator, it reads
——————————————————— where *independent Hamiltonian operator* it reads.
Ime-independent Hamiltonian operator it reads. frequency of oscillation, *q* is an electric charged, where m is the mass of the particle, α and $|E|$ is an electric field. We rewrite Eq. (18) a *p p a p i b i b i b i n i i n i i n i i n i n i n i n i n i n i n i n i n i n i n i n i n i n i n i n i n i n i*

(14)
$$
\hat{H} = \frac{\hat{p}^2(0)}{2m} + \frac{1}{2}m\omega^2 \hat{X}^2(0) - qE\hat{X}(0)
$$
 (19)

م
a despite the fact that the operator and a
despite time dependent. It is matter of choice whether to work with
e calculation of the Hamiltonian operator given by Eq.(18) or by Eq.(19). time dependent. It is matter of choice whether to work with
the Hamiltonian operator given by Eq.(18) or by Eq.(19).
For simplicity, we choose the latter be determined For simplicity, we choose the latter.
As stated in step (i), we start by writing down ^ˆ() ^ˆ () *d Pt X t dt m* ⁼ (20a) *dt m* ⁼ (20a) *dt m* ⁼ (20a) ^ˆ() ^ˆ () *d Pt X t* ^ˆ() ^ˆ () *d Pt X t* $\begin{array}{c} 2.77 & 2.77 \\ \end{array}$ despite the fact that the operator and are explicitly $\frac{1}{2}$ despite the fact that the operator and are explic the fact that the operator and are explicitly the time-independent Hamiltonian operator, it reads to the time*m* For simplicity, we choose the latter.

2 *P qE X X* = +− 2 *P qE X X* 2 *P qE X X* = +− 2 *P qE X X* = +− 2 *P qE X X* =− + (20b) As stated in step (i), we start by writing of
the corresponding Heisenberg equations: $\mathbf{r} = \mathbf{r} \cdot \mathbf{r}$ As stated in step (i), we start by writing down
the corresponding Heisenberg equations: As stated in step (i), we start by writing down
the corresponding Uniophers equations: the corresponding Heisenberg equations:

$$
\langle 0 \rangle \qquad (15a)
$$
\n
$$
\frac{d}{dt} \hat{X}(t) = \frac{\hat{P}(t)}{m}
$$
\n
$$
\langle 20a \rangle
$$
\n
$$
\langle 15b \rangle \qquad (20a)
$$

(15b)
\n
$$
\frac{d}{dt}\hat{P}(t) = -m\omega^2 \hat{X}(t) + qE
$$
\n(20b)

that the usual whose solutions permit us to write for $t = \tau$ that whose solutions permit us to write for $t - \tau$ that whose solutions permit us to write for $t = \tau$ that

$$
\hat{X}(\tau) = \hat{X}(0)\cos\omega\tau + \frac{\hat{P}(0)}{m\omega}\sin\omega\tau - \frac{2qE}{m\omega^2}\sin^2\frac{\omega\tau}{2}
$$
 (21)

 $\frac{1}{100}$
For later convenience, we also write the multiplicative corresponding expression for $\hat{P}(\tau)$:
can be done = −− \overline{F} terms of ˆ *X* () and ˆ **For later convenience, we also write the corresponding expression for** $\hat{P}(\tau)$ **:** corresponding expression for $\mathcal{L}_\mathbf{z}$ For later convenience, we also write the $\frac{1}{2}$ (21) The later concentrate was also with corresponding expression for \overline{z}

done

\n
$$
\hat{P}(\tau) = -m\omega \hat{X}(0) \sin \omega \tau + \hat{P}(0) \cos \omega \tau + \frac{qE}{\omega} \sin \omega \tau
$$
\n(17)

\n(18)

Í ١ To complete step (ii) we need to rewrite $\hat P(0)$ $\frac{1}{1}$ $\frac{w}{z}$ **b** (*m*) (*m* \sim \sim \sim To complete step (ii) we need to rewrite $T(0)$ If we substitute the $\Lambda(\nu)$, which can be done allocally To complete step (ii) we need to rewrite $\hat{P}(0)$
1 terms of $\hat{X}(\tau)$ and $\hat{X}(0)$, which can be done directly
220 **Fo** (24) \hat{P} **To complete step (ii) we need to rewrite** \hat{P} **(i)** $\lim_{x \to \infty} \frac{P(x)}{P(x)} = \lim_{x \to \infty} \frac{P(x)}{P(x)}$ To complete step (ii) we need to rewrite $P(0)$
in terms of $\hat{X}(\tau)$ and $\hat{X}(0)$, which can be done directly $L(23)$. $2x (3)$ and $2x (2)$ from Eq.(21): \equiv q.(21): To complete step (ii) we need to rewrite $\hat{P}(0)$ in terms of $\hat{X}(\tau)$ and $\hat{X}(0)$, which can be done directly 2π $\mathbb{Z}(2)$. $q(x)$ *p* $q(x)$ *p* $\frac{1}{2}$ *p* $\frac{1$

$$
\hat{P}(0) = \frac{m\omega}{\sin \omega \tau} \Big[\hat{X}(\tau) - \hat{X}(0) \cos \omega \tau - \frac{2qE}{m\omega^2} \sin^2 \frac{\omega \tau}{2} \Big] (23) \qquad H(x_b, x_a; \tau) = \frac{\langle x_b, \tau | \hat{H} | x_a, 0 \rangle}{\langle x_b, \tau | x_a, 0 \rangle}
$$

If we substitute this result into Eq. (19), we obtain this result into Eq. (19 $\,$ If we substitute this result into Eq. (19), w If we substitute this result into Eq. (19), we obta ² 2 ² ˆ ˆˆ (0) () (0)cos sin sin ² *^m qE P XX* اس a substitute this result into Eq. (19), w $\frac{1}{2}$ we substitute this result line Eq. (19), we obtain $\ddot{}$

$$
\hat{H} = \frac{m\omega^2}{2\sin^2 \omega \tau} \left[\hat{X}^2(\tau) + \hat{X}^2(0) \cos^2 \omega \tau \right. \\
\left. + \frac{4qEx_0}{m\omega^2} \sin^4 \frac{\omega \tau}{2} \csc^2 \omega \tau - \frac{4qEx_0}{m\omega^2} \sin^2 \omega \tau \right. \\
\left. - \hat{X}(0) \hat{X}(\tau) \cos \omega \tau - \hat{X}(\tau) \hat{X}(0) \cos \omega \tau \right. \\
\left. + \frac{4qE\hat{X}(0)}{m\omega^2} \cos \omega \tau \sin^2 \frac{\omega \tau}{2} \right. \\
\left. - \frac{4qE\hat{X}(0)}{m\omega^2} \cos \omega \tau \sin^2 \frac{\omega \tau}{2} \right. \\
\left. - \frac{4qE\hat{X}(\tau)}{m\omega^2} \sin^2 \frac{\omega \tau}{2} + \frac{4q^2E^2}{m^2\omega^4} \sin^4(\omega \tau/2) \right] \right. \\
\left. + \frac{1}{2}m\omega^2 \hat{X}^2(0) - qE\hat{X}(0) \right. \\
\left. + \frac{1}{2}m\omega^2 \hat{X}^2(0) - qE\hat{X}(0) \right. \\
\left. + \frac{4q^2E^2}{m^2\omega^4} \sin^4 \frac{\omega \tau}{2} \csc^2 \omega \tau' - 2x_b x_a \cot \omega \right. \\
\left. + \frac{4q^2E^2}{m^2\omega^4} \sin^4 \frac{\omega \tau'}{2} \csc^2 \omega \tau' - \frac{4qE(x_b)}{m^2\omega^2} \right]
$$

Note that the third term in Eq. (24) is not written
in the appropriate order. By using the commutation \times sin in the appropriate order. By using the commutation **relation** $\frac{1}{\sqrt{2}}$ using the commutation order. By using the commutation of $\frac{1}{\sqrt{2}}$ using the commutation of $\frac{1}{\sqrt{2}}$ relation e appropriate order. By using the commutation \mathbf{N} m are appropriate that by analysis commonly appropriate order By using the communi ˆˆ ˆˆ (0) () () (0) sin *ⁱ XX XX ^m* = + (26) ² sin / 2 *qE* te order. By using the commutation $\frac{1}{2}$ in the time in Eq. (24) is not
a order. By using the commi e order. By us Note that the third term in Eq. (24) is not written

in the appropriate order. By using the commutation

relation

x sin² $\frac{\omega}{2}$ elation
i \blacksquare relation **and** , ,0 (,)exp ² Note that the third term in Eq. (24) is not wi Note that the third term
*i*ne appropriate order. By
نفاذ Note that the third term in E $\overline{}$ Note that the third term in Eq. Note that the third term in Eq. (24) is not writing the appropriate order. By using the commuta \blacksquare and \blacksquare :q. (24) is not wri sin *ⁱ* Note that the third term in Eq. (24) is not $\overline{\mathcal{O}}$ ation *^P XX X X* $\overline{}$ $\overline{\$ *P* $\frac{1}{2}$ *<i>XX* $\frac{1}{2}$ *XX* $\frac{1}{2$ the appropriate order. By using the cor -7

$$
[\hat{X}(0), \hat{X}(\tau)] = [\hat{X}(0), \hat{X}(0)\cos\omega\tau + \frac{\hat{P}(0)}{m\omega}\sin\omega\tau \qquad \text{evaluate}
$$

$$
-\frac{2qE}{m\omega^2}\sin^2(\omega\tau/2)] \qquad (25)
$$

$$
=\frac{i}{m\omega}\sin\omega\tau \qquad (25)
$$

It follows immediately that It follows immediately that sin *ⁱ* It follows immediately that It follows immediately that *^m H X X XX* It follows immediately that *m* follows immediately that follows immed It follows immediately that $\overline{}$ and $\overline{}$ *<i>i ii ii ii ii*

$$
\hat{X}(0)\hat{X}(\tau) = \hat{X}(\tau)\hat{X}(0) + \frac{i}{m\omega}\sin \omega \tau \tag{26}
$$

bstitute Eq. *('* **If we substitute Eq. (26) into Eq. (24), we obter** the ordered Hamiltonian:
 $\frac{m\omega^2}{2}$ = 5.33 = 0.33 If we substitute Eq. (26) into Eq. (24), we obtain
the ordered Hamiltonian: ˆˆ ˆˆ (0) () () (0) sin *ⁱ XX XX* we substitute Eq. (26) into Eq. (2₁ If we substitute Eq. (26) into Eq. (24), we o If we substitute Eq. (26) into Eq. (24), we
pred Hamiltonian: f we substitute Eq. (26) into Eq. (24), we obtair
he ordered Hamiltonian: 2 2 If we substitute Eq. (26)
ordered Hamiltonian:
 $=\frac{m\omega^2}{2\sin^2(\omega\tau)}[\hat{X}^2(\tau)+\hat{X}^2(0)]$ 4 2 2 2 and the contract of the contra **Dividend Hamiltonian:** 2 4 2

$$
\hat{H}_{ord} = \frac{m\omega^2}{2\sin^2 \omega \tau} \left[\hat{X}^2(\tau) + \hat{X}^2(0) - 2\hat{X}(\tau)\hat{X}(0)\cos \omega \tau \right]
$$

$$
+ \frac{4q^2 E^2}{m^2 \omega^4} \sin^4 \frac{\omega \tau}{2} - \frac{4q E \hat{X}(0)}{m \omega^2} \sin^2 \frac{\omega \tau}{2} \quad (27)
$$

$$
- \frac{4q E \hat{X}(\tau)}{m \omega^2} \sin^2 (\omega \tau / 2) \right] - \frac{i \omega}{2} \cot \omega \tau
$$
Once the Hamiltonian operator is appropriately

Once the Hamiltonian operator is appropri Eq. (13): ^ˆ , .0 (, ;) , .0 *x Hx* Once the Hamiltonian operator is appropriately $\qquad \hat{P}(\tau)$ ordered, we can find the function $H(x_b, x_a; \tau)$ directly from s definition, given by Eq. (13) : definition, given by Eq. (13): its definition, given by Eq. (13): *b a* \sim $\frac{1}{2}$ *b above* the Hamiltonian operator i *Hx x* Once the Hamiltonian operator is appropriately $\hat{P}(\tau) = m\alpha$ its definition, given by Eq. (13): ordered. For ˆ $\frac{d}{dx}$, we can find the function $H(x_{i}, x_{i}; \tau)$ directly \mathcal{L} (27) Once the Hamiltonian operator is appropriately $\hat{P}(\tau) = m\omega \cot \omega \tau$ the contract of the contract of the contract of $\frac{1}{2}$ ordered, we can find the function
its definition, given by Eq. (13) ad, we can find the funct ordered, we can find the function \hat{I}
its definition, given by Eq. (13): nd the funct *b a its definition, given by Eq. (13):* ordered, we can find the fu
its definition, given by Eq. *b* can find the function $H(x_{b}, x_{a}; \tau)$ directly from *b a* efinition, given by Eq. (1

ordered. For ˆ

$$
H(x_b, x_a; \tau) = \frac{\langle x_b, \tau | \hat{H} | x_a, 0 \rangle}{\langle x_b, \tau | x_a, 0 \rangle}
$$

\nobtain
\n
$$
= \frac{m\omega^2}{2} \left[(x_b^2 + x_a^2) \csc^2 \omega \tau - 2x_b x_a \cot \omega \tau \csc \omega \tau \right]
$$
\n
$$
+ \frac{4q^2 E^2}{m^2 \omega^4} \sin^4 \frac{\omega \tau}{2} \csc^2 \omega \tau - \frac{4q E x_a}{m \omega^2} \sin^2 \frac{\omega \tau}{2} \csc^2 \omega \tau
$$
\n
$$
- \frac{4q E x_b}{m \omega^2} \sin^2 \frac{\omega \tau}{2} \csc^2 \omega \tau \right] - \frac{i}{2} \omega \cot \omega \tau
$$
\n(28)

By using Eq. (14). we can express the in the following form: y using Eq. (14), we can expr By using Eq. (14), we can express the pr By using Eq. (14), we can express the propagat − − **24** and the following form:

a $\frac{1}{2}$ $\frac{1}{2}$ 2 2 *m m* By using Eq. (14), we can express the propagator The contract of the contract o \sim \sim 22 2 By using Eq. (14) , we can express the propagator , ,0 (,)exp ² by using Eq. (14), we can express the prop
in the following form:
(24) Eq. (14), we can express the pr

and the following form: B_{tot} is choosing form. \sim \sim By using Eq. (14), we can e owing form: *b*_p following forms: B *y* using Eq. (14), w in the following form: in the following form:
 (24) By using Eq. (14), we can express the By using Eq. (14), we can expres

= −

x x x x

 $\mathcal{L}(\mathcal{L}) = \mathcal{L}(\mathcal{L})$

$$
\begin{aligned}\n\text{(a)} \\
\tau(2) \text{]} \quad \langle x_b, \tau | x_a, 0 \rangle &= C(x_b, x_a) \exp\left\{-\frac{i}{\tau} \int^{\tau} d\tau' \Big[\frac{m\omega^2}{2} \times \Big(\Big(x_b^2 + x_a^2\Big) \csc^2 \omega \tau' - 2x_b x_a \cot \omega \tau' \csc \omega \tau'\Big) \right. \\
&\quad \left. + \frac{4q^2 E^2}{m^2 \omega^4} \sin^4 \frac{\omega \tau'}{2} \csc^2 \omega \tau' - \frac{4qE(x_a + x_b)}{m\omega^2} \right. \\
\text{(29)} \\
\text{not written} \\
\text{mmutation} \quad \times \sin^2 \frac{\omega \tau'}{2} \csc^2 \omega \tau' - \frac{i}{2} \omega \cot \omega \tau' \Big] \} \n\end{aligned}
$$

The integration over τ' in Eq. (2) evaluated: The integration over τ' in Eq. ($\ddot{}$ τ be integration over τ' in Γ *x m* megicine *megicine* The integration over τ' in Eq. (29) can be readily − ⁴ 4() sin csc The integration over τ in Eq. (29) can $\frac{1}{2}$ over τ' in Eq. (29) can *qE x x ^m* The integration over τ' in Eq. (29) can be readily evaluated: (x, y) , (x, y) , (y, y) $\mathbf{t} = \mathbf{t} + \mathbf{t} + \mathbf{t} + \mathbf{t}$ rne integration over *z* in Eq. (29)
conclusted: *z i* me integration over *z'* in Eq. (29) can
b evaluated: 2 2 The integration over τ' e valuated. The integration over τ' in Eq. (29)

(29)

$$
(25)
$$
\n
$$
(25)
$$
\n
$$
(x_b, \tau | x_a, 0) = \frac{C(x_b, x_a)}{\sqrt{\sin \omega \tau}} \exp\left\{\frac{i m \omega}{2 \sin \omega \tau} \left(\left(x_b^2 + x_a^2\right)\right)\right\}
$$
\n
$$
\times \cos \omega \tau 2 - x_b x_a + \frac{i q E}{2 \omega \sin \omega \tau}
$$
\n
$$
\times \left[4\left(x_b + x_a - \frac{q E}{m \omega^2}\right) \sin^2 \frac{\omega \tau}{2}\right]
$$
\n
$$
+\frac{q E \tau}{m \omega} \sin \omega \tau \right]\}
$$
\n
$$
(30)
$$
\n
$$
\times (0) + \frac{i}{m \omega} \sin \omega \tau \quad (26)
$$

where $C(x_k, x_{\alpha})$ is a *m* $\lim_{\alpha \to \infty} \frac{\cos(\alpha \cdot \beta)}{\alpha}$ is an answer, magnetic. where $C(x)$ rotion constant 2 *m* \overline{a} to be determined according to step (iii). where $C(x_b, x_a)$ is an arbitrary integration constant
to be determined according to step (iii) where $C\!\left(x_{_{b\!}}\!,\!x_{_{a\!}}\right)$ is an ar where $C(x_{\scriptscriptstyle b}\hspace{-1mm},\hspace{-1mm}x_{\scriptscriptstyle a}\hspace{-1mm})$ is an arbitra where $C(x_{b\cdot} x_a)$ is an arbitrary inte (,) , ,0 exp sin 2 sin where $C(x_b, x_a)$ is an arbitrary integri-
to be determined according to step (iii). otain where $C(x_{\scriptscriptstyle b}\@ifnextchar{^}{\!\:}{},x_{\scriptscriptstyle a}\@ifnextchar{^}{\!\:}{})$ is an arbitrary into

The determination of $C(x_{\scriptscriptstyle b\hspace{-0.03cm}.\hspace{0.1cm}} x_{\scriptscriptstyle a\hspace{-0.03cm}.\hspace{0.1cm}})$ is don aid of Eqs. (15) and (17). However, the operators $\hat{P}(0)$ and $\hat{P}(\tau)$ in terms of the *A*(*t*) and *A*(*v*), appropriately ordered. For *Y*(*d*) has already been done (see Eq.(23)), and for rias aireauy been done (see Eq.(20)), and ion
Find offer substituting Eq. (23) into Eq. (22) t_{max} and σ $\overline{}$ $\overline{}$ ($\overline{}$) $\overline{}$ ($\overline{}$) $\overline{}$ 27) $\hat{X}(t)$ and $\hat{X}(0)$, appropriately ordered. For find after substituting Eq. (23) into Eq. (2 The determination aid of Eqs. (15) and (17). However, we need to rewrite perators $P(0)$ and $P(T)$ in terms of the operato *Called X* (*C)*, appropriately ordered. For $P(U)$ this task has already been done (see Eq. (23)), and for $\hat{P}(\tau)$ we the directly book done (coo Eq. (23), and a
The determination of (23) into Eq. (22): The determination of (,) *Cx x b a* is done the operators $\hat{P}(0)$ and $\hat{P}(\tau)$ in terms of the operators $rac{\omega \tau}{2}$ (27) $\hat{X}(\tau)$ and $\hat{X}(0)$, appropriately ordered. For $\hat{P}(0)$ this task The determination of $C(x_{_b\!x_{_a\!})$ is done with the σ), appropriately bruefed. For $P(\sigma)$ the set ras aireauy been done (see Eq. (23)), and for P (
Particularly constitution Γ_2 (23) into Γ_2 (23). find after substituting Eq. (23) into Eq. (22) : has already been done (see Eq.(23)), and for $\hat{P}(\tau)$ we The determination of $C(x_{b\cdot}^{}x_a^{})$ is done $\cos \theta \tau$ aid of Eqs. (15) and (17). However, we $\Lambda(t)$ and $\Lambda(0)$, appropriation The distribution of $(366 \text{ Lq} \cdot (20))$, $\mu \omega \tau$ find after substituting Eq. (23) into Eq. has already been done (see Eq Fine determination of $C(x_b, x_a)$
of Eqs. (15) and (17). However, we the operators $\hat{P}(0)$ and $\hat{P}(\tau)$ in terms of the op has already been done (see Eq.(23)), and for \hat{P} The determination of $C(x_b, x_a)$ is done with $\frac{1}{3}$ and $\frac{1}{3}$ begin done (see Eq. (23)), and for and architecture and arbitrary constant co aid of Eqs. (15) and (17). However, we no τ) and $\hat{X}(0)$ (27) $\hat{X}(\tau)$ and $\hat{X}(0)$, appropriately ordered. For $\hat{P}(0)$ the $\omega \tau$ **b** find after substituting Eq. (23) into Eq. (22) l $\overline{A} = (3e^x + 4e^y)$, and $\overline{A} = (3e^y + 4e^y)$ *F P* find after substituting Eq. (23) into Eq. (22):

propriately

\n
$$
\hat{P}(\tau) = m\omega \cot \omega \tau \left[\hat{X}(\tau) - \hat{X}(0) \cos \omega \tau \right]
$$
\nirectly from

\n
$$
-m\omega \hat{X}(0) \sin \omega \tau + \frac{qE}{\omega} \sin \omega \tau
$$
\nThus, ω is

\n
$$
[0, \infty) \times [0, \in
$$

Then, by inserting Eqs. (31) and Then, by inserting Eqs. (31) and
 a) it is not difficult to show that: *P* **Product and the set of the set** α , as not annoal to show that. ω
Then, by inserting Eqs. (31) and (30) into Eq. *Prien, by inserting Eqs. (31) and (*
(15a) it is not difficult to show that: \mathcal{S} into Eq. (23) into Eq. (23) into Eq. (23): \overline{a} *P*
Then, by inserting Eqs. (31) and (30) in $\frac{1}{2}$ into Eq. (23) into Eq. (23) into Eq. (23) into Eq. (22): (15a) it is not difficult to show that:
X Pm XX $\frac{1}{\sqrt{t}}$ **X** (31) I hen, by inserting Eqs. (31 ω
Then, by inserting Eqs. (31) a Then, by inserting Eqs. (31) and
(15a) it is not difficult to show that: Then, by inserting Eqs. (31) and (30) i 21 IS

$$
\frac{\partial C(x_b, x_a)}{\partial x_b} = 0
$$
 work is kn
and i

Analogously, by substituting Eqs.(23) and (30) into Eq. (15b) we have that $\partial C(x_b, x_a)/\partial x_a = 0$. The last two relations to like the last two relations independent of x_b and x_a . In order to determine the value of C, we first take the limit $\tau \to 0^+$ on $\langle x_{b}, \tau | x_{a}, 0 \rangle$. If we use Eq.(30), we find that relations tell us that $C(x_b, x_a) = C$, that is, it is a constant ar
Analogously, by substituting Eqs.(23) and (30) into _{______}_____________________ Eq. (15b) we have that $\partial C(x_b, x_a)/\partial x_a = 0$. The last two m relations tell us that $C(x_b, x_a) = C$, that is, it is a constant and independent of x_b and x_a . In order to determine the value of C, we first take the limit $\tau \to 0^+$ on $\langle x_{i}, \tau | x_a, 0 \rangle$. If we theor (,) ⁰ *b a b x* ⁼ (32) Analogously, by substituting Eqs.(23) and (30) into Analogously, by substituting Eqs.(23) and (30) into
Eq. (15b) we have that $\partial C(x_b x_a)/\partial x_a = 0$. The last two independent of x_b and x_a . In order to determine the value syst
of C, we first take the limit $\tau \to 0^+$ on $\langle x, \tau | x, 0 \rangle$. If we then into
Analogously, by substituting Eqs.(23) and (30) into ^a non-r Analogously, by substituting Eqs.(25) and (30) into the non-r
Eq. (15b) we have that $\partial C(x_b, x_a)/\partial x_a = 0$. The last two meth relations tell us that $C(x_{b}, x_{a}) = C$, that is, it is a constant a independent of x_b and x_a . In order to determine the value ϵ of *C*, we first take the limit $\tau \to 0^+$ on $\langle x_{\nu}, \tau | x_{\nu}, \theta \rangle$. If we the limit $\tau \to 0^+$ on $\langle x_{\nu}, \tau | x_{\nu}, \theta \rangle$. find that (,) ⁰ *b a*

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\lim_{\tau \to 0^+} \langle x_b, \tau | x_a, 0 \rangle = \lim_{\tau \to 0^+} \frac{C}{\sqrt{\omega \tau}} \exp \left[\frac{im}{2 \tau} (x_b - x_a)^2 \right]
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= C \sqrt{\frac{2\pi i}{m\omega}} \delta (x_b - x_a)
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If we compare this result with the initial condition, Eq. (17), we obtain $C = \sqrt{m\omega/2\pi i}$ By substituting this result for *C* into Eq.(30), we obtain the Feynman propagator for the harmonic oscillator: this result for C into Eq. (30), we obtain the desired \overline{I} If we compare this result with the initial
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this result for C into Eq.(30), we obtain the
Feynman propagator for the harmonic oscillate Eq. (17), we obtain $C = \sqrt{m\omega/2\pi i}$. By substituting Depa

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K(x_b, x_a; \tau) = \langle x_b, \tau | x_a, 0 \rangle
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= \sqrt{\frac{m\omega}{2\pi i \hbar \sin \omega \tau}} \exp \{ \frac{im\omega}{2\hbar \sin \omega \tau} \left[(x_b^2 + x_a^2) \cos \omega \tau \right]
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-2x_b x_a \} + \frac{iqE}{2 \omega \sin \omega \tau} \left[4(x_b + x_a - \frac{qE}{m\omega^2}) \sin^2 \frac{\omega \tau}{2} \right]
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+ \frac{qE\tau}{m\omega} \sin \omega \tau \}
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we found the Feynman propagator of a simple harmonic oscillator connected to a constant electric field using Schwinger's method, which is usually quantum field theory but also works well for honquantum mechanical problems, even mough happen very often. Schwinger's method is based on the
solution of the Heisenberg equations for the position erator of the contentions operators. Such solutions are the contention and canonical momentum operators. Such solutions are the content then used to write the ordered Hamiltonian operator of the position operators \hat{X} (0) and \hat{X} (t). The utilization of proper operator ordering, along with subsidiary initial conditions, results in the yield of such a propagator. We found the Feynman propagator of a simple
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quantum mechanical problems, even though they don't
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initial conditions, results in the yield of such a propagator. happen very often. Schwinger's method is based on the solution of the Heisenberg equations for the position and canonical momentum operators. Such solutions are Cohe then used to write the ordered Hamiltonian operator of We found that the propagator obtained is consistent with We found that the propagator obtained is consistent with the one obtained using the Feynman path integral in the solution of the Heisenberg equations for the position
and canonical momentum operators. Such solutions are
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theoretical and computational physics. We hope that this

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The authors would like to acknowledge the

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