สมบัติของตัวดำเนินการ $\mathfrak{R}^a_m$ ในปริภูมิโครงสร้างเล็กสุดที่มีอุดมคติ
Properties of $\mathfrak{R}^a_m$-operator in minimal structure space with an ideal

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บทคัดย่อ
ในบทความนี้ ผู้วิจัยได้นำเสนอเซตเปิดแบบ δ-m เซตเปิดแบบ a-m ฟังก์ชันแบบ δ-m-$local$ ตัวดำเนินการแบบ R^a_m บนปริภูมิโครงสร้างเล็กสุดที่มีอุดมคติพร้อมทั้งศึกษาสมบัติของฟังก์ชัน และตัวดำเนินการนี้

คำสำคัญ: เซตเปิดแบบ δ-m เซตเปิดแบบ a-m ฟังก์ชันแบบ δ-m-$local$ ตัวดำเนินการแบบ R^a_m ปริภูมิโครงสร้างเล็กสุดที่มีอุดมคติ

Abstract
In this article, the concepts of δ-m-open sets, a-m-open sets in a minimal structure space with an ideal are introduced. In addition, we present an a-m-$local$ function and an R^a_m-operator in a minimal structure space with an ideal. We studied the properties of the function and this operator.

Keywords: δ-m-open sets, a-m-open sets, δ-m-$local$ functions, R^a_m-operator, a minimal structure space with an ideal.

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Introduction

In 1945, Vaidyanathaswamy (1945) defined a local function in an ideal topological space and studied some properties of this function. In 1996, Maki, Umehara and Noiri (1996) defined a minimal structure and studied some properties of this structure. In 2014, Al-Omeri *et al*. (2014) defined an *a-local* function in an ideal topological space and also studied some properties of an *a-local* function. Later in 2016, Al-Omeri *et al*. (2016) defined an R_a*-operator* in an ideal topological space and studied some properties of this operator. In this article, we introduce the concepts of δ*-m-open* sets and δ*-m-open* sets in a minimal structure space with an ideal and study some fundamental properties. Moreover, we introduce the notions of δ*-m-local* functions and R^a_m*-operators* in minimal structure spaces, along with studying some properties related to an δ*-m-local* function and an R^a_m*-operator* defined above.

Preliminaries

Definition 2.1[5] Let X be a nonempty set and $P(X)$ the power set of X. A subfamily m of $P(X)$ is called a minimal structure (briefly MS) on X if $\emptyset \in m$ and $X \in m$.

By (X,m) we denote a nonempty set X with a minimal structure m on X and it is called a minimal structure space. Each member of m is said to be *m-open* and the complement of *m-open* is said to be *m-closed*.

Definition 2.2 (Noiri & Popa, 2009) Let (X,m) be a minimal structure space and $A \subseteq X$. The *m-closure* of A, denoted by $CI_m(A)$ and the *m-interior* of A, denoted by $Int_m(A)$, are defined as follows ;

1) $CI_m(A)=\cap\{F:A\subseteq F, X\setminus F\in m\}$,

2) $Int_m(A)=\cup\{U:U\subseteq A, U\in m\}$.

Lemma 2.3 (Maki & Gani, 1999) Let (X,m) be a minimal structure space and $A,B\subseteq X$, the following properties hold ;

(1) $CI_m(X\setminus A)=X\setminus Int_m(A)$ and $Int_m(X\setminus A)=X\setminus CI_m(A)$.

(2) If $X\setminus A\in m$, then $CI_m(A)=A$ and if $A\in m$, then $Int_m(A)=A$.

(3) $CI_m(\emptyset)=\emptyset$, $CI_m(X)=X$, $Int_m(\emptyset)=\emptyset$, and $Int_m(X)=X$.

(4) If $A\subseteq B$, then $CI_m(A)\subseteq CI_m(B)$ and $Int_m(A)\subseteq Int_m(B)$.

(5) $A\subseteq CI_m(A)$ and $Int_m(A)\subseteq A$.

(6) $CI_m(CI_m(A)) = CI_m(A)$ and $Int_m(Int_m(A)) = Int_m(A)$.

Lemma 2.4 (Maki & Gani, 1999) Let (X,m) be a minimal structure space and $A\subseteq X, x\in X$. Then $x\in CI_m(A)$ if and only if $U\cap A\neq\emptyset)$ for every an *m-open* set U containing X.

Definition 2.5 (Rosas *et al*., 2009) Let (X,m) be a minimal structure space and $A\subseteq X$.

(1) A is called *m-regular* open if $A=Int_m(CI_m(A))$

(2) A is called *m-regular* closed if $X\setminus A$ is *m-regular* open.

The family of all *m-regular* open sets of X is denoted by $r(m)$ and the family of all *m-regular* closed sets of X is denoted by $rc(m)$.

Definition 2.6 (Ozbakir & Yildirim, 2009) An ideal $\mathscr{I}$ on a minimal structure space (X,m) is a nonempty collection of subsets of X which satisfies the following properties ;

(1) $A\in\mathscr{I}$ and $B\subseteq A$ implies $B\in\mathscr{I}$ (heredity),

(2) $A\in\mathscr{I}$ and $B\in\mathscr{I}$ implies $A\cup B\in\mathscr{I}$ (finite additivity).

The set $\mathscr{I}$ together with a minimal structure space (X,m) is called a minimal structure space with an ideal, denoted by $(X,m,\mathscr{I})$.

Main Results

Definition 3.1 Let (X,m) be a minimal structure space. A subset A is said to be δ*-m-open* if for each $X\in A$ there exists an *m-regular* open set G such that $X\in G\subseteq A$. The complement of δ*-m-open* set is called δ*-m-closed*. The family of all δ*-m-closed* sets of X, denoted by $\delta C_m(X)$.

Theorem 3.2 Let (X,m) be a minimal structure space and $A\subseteq X$. The arbitrary union of δ*-m-open* sets is a δ*-m-open* set.

Proof Let B_α be a δ*-m-open* set for all $\alpha\in J$ where J is an index set and let $x\in\bigcup_{\alpha\in J}B_\alpha$. There exists $\beta\in J$ such that $x\in B_\alpha$. Since B_β is δ*-m-open*, there exists an *m-regular* open set G_β such that $X\in G_\beta\subseteq B_\beta$. Then $x\in G_\beta\subseteq B_\beta\subseteq\bigcup_{\alpha\in J}B_\alpha$. Therefore $\bigcup_{\alpha\in J}B_\alpha$ is δ*-m-open*.

Definition 3.3 Let (X,m) be a minimal structure space and $A\subseteq X$. A point $x\in X$ is called a δ-*m-cluster* point of A if $U\cap A\neq\emptyset$ for each *m-regular* open set U containing X.

Definition 3.4 Let (X,m) be a minimal structure space and $A\subseteq X$. The set of all δ-*m-cluster* points of A is called δ-*m-closure* of A and is denoted by $C_{\delta m}(A)$ and the union *m-regular* open sets contained in A is called the δ-*m*-interior of A, denoted by $I_{\delta m}(A)$.

Theorem 3.5 Let (X,m) be a minimal structure space and $A\subseteq X$. Then A is δ-*m-open* if and only if $I_{\delta m}(A)=A$.

Proof ($\Rightarrow$) Suppose that A is δ-*m-open*. By definition of δ-*m-interior*, $I_{\delta m}(A)=A$. Let $x\in A$. Since A is δ-*m-open*, there exists an *m-regular* open set O such that $x\in O\subseteq A$. This implies that $x\in I_{\delta m}(A)$. Then $A\subseteq I_{\delta m}(A)$. Hence $A=I_{\delta m}(A)=A$. ($\Leftarrow$) It follows from Theorem 3.2.

Theorem 3.6 Let (X,m) be a minimal structure space and $A,B\subseteq X$. The following property hold ;

(1) If $A\subseteq B$, then $I_{\delta m}(A)\subseteq I_{\delta m}(B)$,

(2) If $A\subseteq B$, then $C_{\delta m}(A)\subseteq C_{\delta m}(B)$.

Proof (1) Assume that $A\subseteq B$ and $x\in I_{\delta m}(A)$. Then, there exists an *m-regular* open set G such that $x\in G\subseteq A$. Since $A\subseteq B$, we have $x\in G\subseteq A\subseteq B$. This implies that $x\in I_{\delta m}(B)$. Hence $I_{\delta m}(A)\subseteq I_{\delta m}(B)$.

(2) Let $A\subseteq B$. Assume that $x\notin C_{\delta m}(B)$. Then there exists an *m-regular* open set U containing X such that $U\cap B=\emptyset$. Since $A\subseteq B$, we have $U\cap A\subseteq U\cap B=\emptyset$. Thus $x\notin C_{\delta m}(A)$. Therefore $C_{\delta m}(A)\subseteq C_{\delta m}(B)$.

Theorem 3.7 Let (X,m) be a minimal structure space and $A\subseteq X$. The following properties hold ;

(1) $C_{\delta m}(A) = X\setminus I_{\delta m}(X\setminus A)$,

(2) $I_{\delta m}(A) = X\setminus C_{\delta m}(X\setminus A)$.

Proof (1) We will show that $C_{\delta m}(A) = X\setminus I_{\delta m}(X\setminus A)$ by contrapositive. Assume that $x\notin X\setminus I_{\delta m}(X\setminus A)$. We get that $X\setminus I_{\delta m}(X\setminus A)$. So there exists an *m-regular* open set G such that $x\in G\subseteq X\setminus A$. Then $G\cap A=\emptyset$ and $x\notin C_{\delta m}(A)$. Thus $C_{\delta m}(A)\subseteq X\setminus I_{\delta m}(X\setminus A)$.

Next, we show that $X\setminus I_{\delta m}(X\setminus A)\subseteq C_{\delta m}(A)$ by contrapositive. Assume that $x\notin C_{\delta m}(A)$. Then x is not a δ-*m-cluster* point of A. There exists an *m-regular* open set G containing x such that $G\cap A=\emptyset$. So $x\in G\subseteq X\setminus A$ and we get that $x\in I_{\delta m}(X\setminus A)$. Hence $x\notin X\setminus I_{\delta m}(X\setminus A)$. Thus $X\setminus I_{\delta m}(X\setminus A)\subseteq C_{\delta m}(A)$.

(2) Since $X\setminus A\subseteq X$, we have $C_{\delta m}(X\setminus A) = X\setminus I_{\delta m}(X\setminus(X\setminus A))$ by (1) and we get $C_{\delta m}(X\setminus A) = X\setminus I_{\delta m}(A)$. Therefore $I_{\delta m}(X\setminus A) = X\setminus C_{\delta m}(X\setminus A)$.

Definition 3.8 Let (X,m) be a minimal structure space and $A\subseteq X$.

(1) A is called *a-m-open* if $A\subseteq Int_m(CI_m(I_{\delta m}(A)))$. The family of all *a-m-open* sets of X is denoted by $\mathscr{M}^a$.

(2) A is called *a-m-closed* if $CI_{\delta m}(Int_m(C_{\delta m}(A)))\subseteq A$.

Theorem 3.9 Let (X,m) be a minimal structure space and $A\subseteq X$. Then A is *a-m-open* if and only if $X\setminus A$ is *a-m-closed*.

Proof Assume that A is *a-m-open*. Then $A\subseteq Int_m(CI_m(I_{\delta m}(A)))$. and $X\setminus A\supseteq X\setminus(Int_m(CI_m(I_{\delta m}(A))))$. By Lemma 2.3 and Theorem 3.7, $X\setminus A\supseteq CI_m(Int_m(C_{\delta m}(X\setminus A)))$. Therefore, $X\setminus A$ is *a-m-closed*.

Conversely, assume that $X\setminus A$ is *a-m-closed*. Then $CI_m(Int_m(C_{\delta m}(X\setminus A)))\subseteq X\setminus A$ and $X\setminus CI_m(Int_m(C_{\delta m}(X\setminus A)))\supseteq X\setminus(X\setminus A)$. By Lemma 2.3 and Theorem 3.7, $Int_m(CI_m(I_{\delta m}(A)))\supseteq A$. Hence A is *a-m-open*.

Example 3.10 Let $X = \{a, b, c, d\}$ with a minimal structure $m = \{\emptyset, \{a,b\}, \{b,c\}, \{c,d\}, \{a,d\}, x\}$. Then $r(m) = \{\emptyset, \{a,b\}, \{a,d\}, \{b,c\}, \{c,d\}, x\}$, and $\delta O_m(x) = \{\emptyset, \{a,b\}, \{a,d\}, \{b,c\}, \{c,d\}, \{a,b,c\}, \{a,b,d\}, \{a,c,d\}, \{b,c,d\}, x\}$, $\mathscr{M}^a = \{\emptyset, \{a,b\}, \{a,d\}, \{b,c\}, \{c,d\}, \{a,b,c\}, \{a,b,d\}, \{a,c,d\}, \{b,c,d\}, x\}$. In this example $\{a,b\},\{a,d\}\in\mathscr{M}^a$ but $\{a,b\}\cap\{a,d\} = \{a\}\notin\mathscr{M}^a$, that means $\mathscr{M}^a$ does not have the property that any finite intersection of *a-m-open* sets is *a-m-open*.

Definition 3.11 Let (X,m) be a minimal structure space and $A\subseteq X$. The *a-m-closure* of A, denoted by $aC_m(A)$ and the *a-m-interior* of A, denoted by $aI_m(A)$, are defined as follows ;

(1) $aC_m(A)=\cap\{F:X\setminus F\in\mathscr{M}^a$ and $A\subseteq F\}$,

(2) $aI_m(A)=\cup\{U:U\in\mathscr{M}^a$ and $U\subseteq A\}$.

Theorem 3.12 Let (X,m) be a minimal structure space and $A\subseteq X$, $x\in X$, Then $x\in aC_m(A)$ if and only if $U\cap A\neq\emptyset$ for every *a-m-open* set U containing x.

Proof ($\Rightarrow$) Suppose that there exists an *a-m-open* set U containing x such that $U\cap A=\emptyset$. So $A\subseteq X\setminus U$ and $X\setminus U$ is *a-m-closed*. Since $aC_m(A)$ is the intersection of all *a-m-closed* sets containing A, $aC_m(A) \subseteq X \setminus U$. Since $x \notin X \setminus U$, we have $x \notin aC_m(A)$.

($\Leftarrow$) Assume that $x \notin aC_m(A)$. Then there exists an *a-m-closed* set F such that $A \subseteq F$ and $x \notin F$. Choose $U = X \setminus F$. Then U is *a-m-open* and $x \in X \setminus F = U$. Moreover, $U \cap A \subseteq (X \setminus F) \cap F = \emptyset$.

Theorem 3.13 Let (X,m) be a minimal structure space and $A, B \subseteq X$. The following properties hold ;

(1) If $A \subseteq B$, then $aC_m(A) \subseteq aC_m(B)$.

(2) If $A \subseteq B$, then $aI_m(A) \subseteq aI_m(B)$.

Proof (1) Assume that $A \subseteq B$ and $x \notin aC_m(B)$. Then there exists an *a-m-open* set U containing x such that $U \cap F = \emptyset$. Since $A \subseteq B$, $U \cap A = \emptyset$. Hence $x \notin aC_m(A)$.

(2) Let $A \subseteq B$ and $x \in aI_m(A)$. Then there exists an a-m-open set U such that $x \in U \subseteq A$. Since $A \subseteq B$, $x \in U \subseteq B$. Therefore $x \in aI_m(B)$.

Proposition 3.14 Let (X,m) be a minimal structure space. Then $\emptyset \in \mathcal{M}^a$ and $X \in \mathcal{M}^a$.

Proof Since $\emptyset \subseteq Int_m(CI_m(I_{\delta m}(\emptyset)))$, $\emptyset$ is *a-m-open*, and so $\emptyset \in \mathcal{M}^a$. Clearly $X = Int_m(CI_m(X))$, so X is an *m-regular* open. Then X is *δ-m-open*, that is $I_{\delta m}(X) = X$, and so $X \subseteq Int_m(CI_m(I_{\delta m}(X)))$. Therefore $X \in \mathcal{M}^a$.

Theorem 3.15 Let (X,m) be a minimal structure space. Then the arbitrary union of elements of $\mathcal{M}^a$ belongs to $\mathcal{M}^a$.

Proof Let V_α be *a-m-open* for all $\alpha \in J$ and $G = \bigcup_{\alpha \in J} V_\alpha$. Then $V_\alpha \subseteq Int_m(CI_m(I_{\delta m}(V_\alpha)))$ for all $\alpha \in J$. Since $V_\alpha \subseteq G$, it follows that $I_{\delta m}(V_\alpha) \subseteq I_{\delta m}(G)$ and so $CI_m(I_{\delta m}(V_\alpha)) \subseteq CI_m(I_{\delta m}(G))$. Then $Int_m(CI_m(I_{\delta m}(V_\alpha))) \subseteq Int_m(CI_m(I_{\delta m}(G)))$. This implies that $V_\alpha \subseteq Int_m(CI_m(I_{\delta m}(G)))$ for all $\alpha \in J$. Thus $\bigcup_{\alpha \in J} V_\alpha \subseteq Int_m(CI_m(I_{\delta m}(G)))$. Therefore $G \subseteq Int_m(CI_m(I_{\delta m}(G)))$.

Corollary 3.16 Let (X,m) be a minimal structure space. Then the arbitrary intersection of *a-m-closed* sets is an *a-m-closed* set.

Proof Let G_α be *a-m-closed* for all $\alpha \in J$. Then $X \setminus G_\alpha$ is *a-m-open* and so $\bigcup_{\alpha \in J}(X \setminus G_\alpha)$ is *a-m-open*. Since $X \setminus \bigcap_{\alpha \in J} G_\alpha = \bigcup_{\alpha \in j}(X \setminus G_\alpha)$, $\bigcap_{\alpha \in J} G_\alpha$ is *a-m-closed*.

Remark 3.17 In a minimal structure space, by Corollary 3.16, $aC_m(A)$ is *a-m-closed*.

Theorem 3.18 Let (X,m) be a minimal structure space and $A \subseteq X$. The following properties hold ;

(1) $aC_m(aC_m(A)) = aC_m(A)$,

(2) $aI_m(aI_m(A)) = aI_m(A)$.

Proof (1) Clearly $aC_m(A) \subseteq aC_m(aC_m(A))$. Since $aC_m(A)$ is *a-m-closed*, $aC_m(aC_m(A)) \subseteq aC_m(A)$.Therefore $aC_m(aC_m(A)) = aC_m(A)$.

(2) Clearly $aI_m(aI_m(A)) = aI_m(A)$. Since $aI_m(A)$ *a-m-open*, $aI_m(A) \subseteq aI_m(aI_m(A))$. Therefore $aI_m(aI_m(A)) = aI_m(A)$.

Let $(X,m,\mathcal{I})$ be a minimal structure space with an ideal. For each $x \in X$, let $\mathcal{M}^a(x) = \{U : x \in U, U \in \mathcal{M}^a\}$ be the family of all *a-m-open* sets that contain x.

Definition 3.19 Let $(X,m,\mathcal{I})$ be a minimal structure space with an ideal and $A \subseteq X$. Then $A^{a^*}_m(\mathcal{I},m) = \{x \in X : U \cap A \notin \mathcal{I}$, for every $U \in \mathcal{M}^a(x)\}$ is called a-m *-local* function of A with respect to $\mathcal{I}$ and m. We denote simply $A^{a^*}_m$ for $A^{a^*}_m(\mathcal{I},m)$.

Remark 3.20 The minimal ideal is $\{\emptyset\}$ and the maximal ideal is $P(x)$ in any minimal structure space with an ideal $(X,m,\mathcal{I})$. It can be deduced that $A^{a^*}_m(\{\emptyset\},m) = aC_m(A)$ and $A^{a^*}_m(P(X),m) = \emptyset$ for every $A \subseteq X$.

Remark 3.21 In general, $A \not\subset A^{a^*}_m$ and $A^{a^*}_m \not\subset A$. The next example shows that $A \not\subset A^{a^*}_m$.

Example 3.22 Let $X = \{a,b,c,d\}$ with a minimal structure $m = \{\emptyset, \{a,b\}, \{b,c\}, \{c,d\}, \{a,d\}, X\}$ and $\mathcal{I} = \{\emptyset, \{a\}, \{b\}, \{a,b\}\}$, $A = \{a,b\}$. Then $\mathcal{M}^a = \{\emptyset, \{a,b\}, \{a,d\}, \{b,c\}, \{c,d\}, \{a,b,c\}, \{a,b,d\}, \{a,c,d\}, \{b,c,d\}, X\}$ and $A^{a^*}_m = \emptyset$.

Theorem 3.23 Let $(X,m,\mathcal{I})$ be a minimal structure space with an ideal and $A, B \subseteq X$. The following properties hold ;

(1) $(\emptyset)^{a^*}_m = \emptyset$.]

(2) If $A \subseteq B$, then $A^{a^*}_m \subseteq B^{a^*}_m$.

(3) $(A^{a^*}_m)^{a^*}_m \subseteq A^{a^*}_m$.

(4) $A^{a^*}_m \cup B^{a^*}_m \subseteq (A \cup B)^{a^*}_m$.

(5) $(A \cap B)^{a^*}_m \subseteq A^{a^*}_m \cap B^{a^*}_m$.

(6) $(A \setminus B)^{a^*}_m \setminus (B)^{a^*}_m \subseteq A^{a^*}_m \setminus B^{a^*}_m$.

Proof (1) Assume $(\emptyset)^{a^*}_m \neq \emptyset$. Then there exists $x \in (\emptyset)^{a^*}_m$. Since $X \in \mathcal{M}^a(X)$, $X \cap \emptyset \notin \mathcal{I}$. It contradicts with $X \cap \emptyset = \emptyset \notin \mathcal{I}$. Therefore $(\emptyset)^{a^*}_m = \emptyset$.

(2) Assume that $A \subseteq B$. We will show that $A^{a^*}_m \subseteq B^{a^*}_m$ by contrapositive. Suppose that $x \notin B^{a^*}_m$. Then there exists $U \in \mathcal{M}^a(X)$ such that $U \cap B \in \mathcal{I}$. From $A \subseteq B$ and the property of $\mathcal{I}$, $U \cap A \in \mathcal{I}$. Therefore $x \notin A^{a^*}_m$.

(3) Assume that $x \in (A^{a^*}_m)^{a^*}_m$, and $U \in \mathcal{M}^a(X)$. Then $A^{a^*}_m \cap U \notin \mathcal{I}$ and so $A^{a^*}_m \cap U \neq \emptyset$. Thus there exists $y \in A^{a^*}_m \cap U$, and so $y \in U \in \mathcal{M}^a(y)$. This implies that $A \cap U \notin \mathcal{I}$. Therefore $x \in A^{a^*}_m$.

(4) Since $A \subseteq A \cup B$ and $B \subseteq A \cup B$, by (2) $A^{a^*}_m \subseteq (A \cup B)^{a^*}_m$ and $B^{a^*}_m \subseteq (A \cup B)^{a^*}_m$. So $A^{a^*}_m \cup B^{a^*}_m \subseteq (A \cup B)^{a^*}_m$.

(5) Since $A \cap B \subseteq A$ and $A \cap B \subseteq B$, by (2) $(A \cap B)^{a^*}_m \subseteq A^{a^*}_m$ and $(A \cap B)^{a^*}_m \subseteq B^{a^*}_m$. So $(A \cap B)^{a^*}_m \subseteq A^{a^*}_m \cap B^{a^*}_m$.

(6) Since $A \setminus B \subseteq A$, by (2) $(A \setminus B)^{a^*}_m \subseteq A^{a^*}_m$. So $(A \setminus B)^{a^*}_m \setminus B^{a^*}_m \subseteq A^{a^*}_m \setminus B^{a^*}_m$.

Theorem 3.24 Let (X,m) be a minimal structure space and $\mathcal{I}$, $\mathcal{J}$ are ideals on X where $\mathcal{I} \subseteq \mathcal{J}$. Then $A^{a^*}_m(\mathcal{J},m) \subseteq A^{a^*}_m(\mathcal{I},m)$ for all $A \subseteq X$.

Proof Let $A \subseteq X$. Assume that $x \in A^{a^*}_m(\mathcal{J},m)$. Then $U \cap A \notin \mathcal{J}$ for every $U \in \mathcal{M}^a(x)$. Since $\mathcal{I} \subseteq \mathcal{J}$, $U \cap A \notin \mathcal{I}$ for every $U \in \mathcal{M}^a(x)$. Thus $x \in A^{a^*}_m(\mathcal{I}, m)$. Hence $A^{a^*}_m(\mathcal{J}, m) \subseteq A^{a^*}_m(\mathcal{I}, m)$.

Theorem 3.25 Let $(X,m,\mathcal{I})$ be a minimal structure space with an ideal and $A \subseteq X$. The following properties hold ;

(1) $A^{a^*}_m \subseteq aC_m(A)$,

(2) $A^{a^*}_m = aC_m(A)$, (*i.e.*, $A^{a^*}_m$ is an *a-m-closed* subset).

Proof (1) Assume that $x \notin aC_m(A)$. Then there exists an *a-m-closed* set F such that $A \subseteq F$ and $x \notin F$. Thus $x \in X \setminus F$, and so $X \setminus F \in \mathcal{M}^a(x)$. Hence $(X \setminus F) \cap A = \emptyset \in \mathcal{I}$, and so $x \notin A^{a^*}_m$. This implies that $A^{a^*}_m \subseteq aC_m(A)$.

(2) It is clear that $A^{a^*}_m \subseteq aC_m(A^{a^*}_m)$. Next, we will prove that $aC_m(A^{a^*}_m) \subseteq A^{a^*}_m$. Let $x \in aC_m(A^{a^*}_m)$ and $U \in \mathcal{M}^a(x)$. Then $A^{a^*}_m \cap U \neq \emptyset$. Therefore there exists $y \in A^{a^*}_m \cap U$, so $U \in \mathcal{M}^a(y)$. Since $y \in A^{a^*}_m$, $A \cap U \notin \mathcal{I}$, and so $x \in A^{a^*}_m$. Then $A^{a^*}_m = aC_m(A^{a^*}_m)$.

Theorem 3.26 Let $(X,m,\mathcal{I})$ be a minimal structure space with an ideal and $A \subseteq X$. The following properties hold ;

(1) If $A \in \mathcal{I}$, then $A^{a^*}_m = \emptyset$.

(2) If $U \in \mathcal{I}$, then $A^{a^*}_m = (A \cup U)^{a^*}_m$.

(3) If $U \in \mathcal{I}$, then $A^{a^*}_m = (A \setminus U)^{a^*}_m$.

Proof (1) Assume that $A^{a^*}_m \neq \emptyset$. Then there exists $x \in A^{a^*}_m$. Since $X \in \mathcal{M}^a(x)$, $A = X \cap A \notin \mathcal{I}$.

(2) Assume that $U \in \mathcal{I}$. Since $A \subseteq A \cup U$ by Theorem 3.23(2), we get $A^{a^*}_m \subseteq (A \cup U)^{a^*}_m$. Next, we will prove that $(A \cup U)^{a^*}_m \subseteq A^{a^*}_m$ by contrapositive. Suppose that $x \notin A^{a^*}_m$. Then there exists $V \in \mathcal{M}^a(x)$ such that $A \cap V \in \mathcal{I}$. Since $(A \cup U) \cap V = (A \cap V) \cup (U \cap V) \in \mathcal{I}$, $(A \cup U) \cap V \in \mathcal{I}$. Therefore $x \notin (A \cap U)^{a^*}_m$.

(3) Assume that $U \in \mathcal{I}$. Since $A^{a^*}_m = (A \cap X)^{a^*}_m = (A \cap ((X \setminus U) \cup U))^{a^*}_m = ((A \setminus U) \cup (A \cup U))^{a^*}_m$ and $A \cap U \subseteq U \in \mathcal{I}$, by (2) $A^{a^*}_m = (A \setminus U)^{a^*}_m$.

Definition 3.27 Let $(X,m,\mathcal{I})$ be a minimal structure with an ideal. An operator $\mathfrak{R}^a_m : P(X) \to P(X)$ is defined as follows ; for every $A \in P(X)$, $\mathfrak{R}^a_m(A) = \{x \in X :$ there exists $U \in \mathcal{M}^a(x)$ such that $U \setminus A \in \mathcal{I}\}$.

Theorem 3.28 Let $(X,m,\mathcal{I})$ be a minimal structure space with an ideal and $A \in P(X)$. Then $\mathfrak{R}^a_m(A) = X \setminus (X \setminus A)^{a^*}_m$.

Proof Let $x \in \mathfrak{R}^a_m(A)$. Then there exists an *a-m-open* set U containing x such that $U \setminus A \in \mathcal{I}$. Thus $U \cap (X \setminus A) \in \mathcal{I}$. So $x \notin (X \setminus A)^{a^*}_m$ and hence $x \in X \setminus (X \setminus A)^{a^*}_m$. Therefore $\mathfrak{R}^a_m(A) \subseteq X \setminus (X \setminus A)^{a^*}_m$.

For the reverse inclusion, let $x \in X \setminus (X \setminus A)^{a^*}_m$. Then $x \notin (X \setminus A)^{a^*}_m$. Thus there exists an *a-m-open* set U containing x such that $U \cap (X \setminus A) \in \mathcal{I}$. This implies that $U \setminus A \in \mathcal{I}$. Hence $x \in \mathfrak{R}^a_m(A)$. So $X \setminus (X \setminus A)^{a^*}_m \subseteq \mathfrak{R}^a_m(A)$. Therefore $\mathfrak{R}^a_m(A) = X \setminus (X \setminus A)^{a^*}_m$.

Example 3.29 Let $X = \{a,b,c,d\}$ with a minimal structure $m = \{\emptyset, \{a,b\}, \{b,c\}, \{c,d\}, \{a,d\}, X\}$ and $\mathcal{I} = \{\emptyset, \{a\}, \{b\}, \{a,b\}\}$, $A = \{a,b\}$. Then $= \{\emptyset, \{a,b\}, \{a,d\}, \{b,c\}, \{c,d\}, \{a,b,c\}, \{a,b,d\}, \{a,c,d\}, \{b,c,d\}, X\}$ and $\mathfrak{R}^a_m(A) = \{a,b\}$.

Theorem 3.30 Let $(X,m,\mathcal{I})$ be a minimal structure space with an ideal, and $A \subseteq X$. Then $\mathfrak{R}^a_m(A)$ is *a-m-open*.

Proof We know that $\mathfrak{R}^a_m(A)=X\setminus(X\setminus A)^{a^*}_m$ and $(X\setminus A)^{a^*}_m$ is *a-m-closed*. Therefore $\mathfrak{R}^a_m(A)$ is *a-m-open*.

Theorem 3.31 Let $(X,m,\mathscr{I})$ be a minimal structure space with an ideal and $A,B\subseteq X$. Then the following properties hold ;

(1) If $A\subseteq B$, then $\mathfrak{R}^a_m(A)\subseteq\mathfrak{R}^a_m(B)$.

(2) If $A\subseteq B$, then $\mathfrak{R}^a_m(A\cap B)\subseteq\mathfrak{R}^a_m(A)\cap\mathfrak{R}^a_m(B)$.

(3) If $A\subseteq B$, then $\mathfrak{R}^a_m(A)\cup\mathfrak{R}^a_m(B)\subseteq\mathfrak{R}^a_m(A\cup B)$.

(4) If $A\in\mathscr{M}^a$, then $A\subseteq\mathfrak{R}^a_m(A)$.

(5) If $A\subseteq B$, then $\mathfrak{R}^a_m(A)\subseteq\mathfrak{R}^a_m(\mathfrak{R}^a_m(A))$.

Proof (1) Assume that $A\subseteq B$. Then $X\setminus B\subseteq X\setminus A$. By Theorem 3.23(2), $(X\setminus B)^{a^*}_m\subseteq(X\setminus A)^{a^*}_m$ and hence $X\setminus(X\setminus A)^{a^*}_m\subseteq X\setminus(X\setminus B)^{a^*}_m$. Therefore $\mathfrak{R}^a_m(A)\subseteq\mathfrak{R}^a_m(B)$.

(2) Since $A\cap B\subseteq A$ and $A\cap B\subseteq B$, $\mathfrak{R}^a_m(A\cap B)\subseteq\mathfrak{R}^a_m(A)$ and $\mathfrak{R}^a_m(A\cap B)\subseteq\mathfrak{R}^a_m(B)$. Therefore $\mathfrak{R}^a_m(A\cap B)\subseteq\mathfrak{R}^a_m(A)\cap\mathfrak{R}^a_m(B)$.]

(3) Since $A\subseteq A\cup B$ and $B\subseteq A\cup B$, $\mathfrak{R}^a_m(A)\subseteq\mathfrak{R}^a_m(A\cup B)$ and $\mathfrak{R}^a_m(B)\subseteq\mathfrak{R}^a_m(A\cup B)$. Therefore $\mathfrak{R}^a_m(A)\cup\mathfrak{R}^a_m(B)\subseteq\mathfrak{R}^a_m(A\cup B)$.

(4) Assume that $A\in\mathscr{M}^a$. Then $X\setminus A$ is *a-m-closed*. By Theorem 3.25(1), we get that $(X\setminus A)^{a^*}_m\subseteq aC_m(X\setminus A)=X\setminus A$. Therefore $A=X\setminus(X\setminus A)\subseteq X(X\setminus A)^{a^*}_m=\mathfrak{R}^a_m(A)$.

(5) By Theorem 3.30, we get that $\mathfrak{R}^a_m(A)$ is *a-m-open*. By (4), we get that $\mathfrak{R}^a_m(A)\subseteq\mathfrak{R}^a_m(\mathfrak{R}^a_m(A))$.

Theorem 3.32 Let $(X,m,\mathscr{I})$ be a minimal structure space with an ideal and $A, B, U\subseteq X$. Then the following properties hold ;

(1) If $U\in\mathscr{I}$, then $\mathfrak{R}^a_m(A\setminus U)=\mathfrak{R}^a_m(A)$.

(2) If $U\in\mathscr{I}$, then $\mathfrak{R}^a_m(A\cup U)=\mathfrak{R}^a_m(A)$.

(3) If $(A\setminus B)\cup(B\setminus A)\in\mathscr{I}$, then $\mathfrak{R}^a_m(A)=\mathfrak{R}^a_m(B)$.

(4) If $A\in\mathscr{I}$, then $\mathfrak{R}^a_m(A)=X\setminus X^{a^*}_m$.

Proof (1) Assume that $A\subseteq X$, $U\in\mathscr{I}$. By Theorem 3.26(2) and 3.28, we have $\mathfrak{R}^a_m(A\setminus U)=X\setminus(X\setminus(A\setminus U))^{a^*}_m$ $=X\setminus((X\setminus A)\cup U))^{a^*}_m$ $=X\setminus(X\setminus A)^{a^*}_m$. Therefore $\mathfrak{R}^a_m(A\setminus U)=\mathfrak{R}^a_m(A)$.

(2) Assume that $U\in\mathscr{I}$. By Theorem 3.26(3), we have $\mathfrak{R}^a_m(A\cup U)=X\setminus(X\setminus(A\cup U))^{a^*}_m$ $=X\setminus((X\setminus A)\setminus U)^{a^*}_m=X\setminus(X\setminus A)^{a^*}_m=\mathfrak{R}^a_m(A)$.

(3) Assume that $(A\setminus B)\cup(B\setminus A)\in\mathscr{I}$. Thus

$$\begin{aligned}\mathfrak{R}^a_m(A)&=\mathfrak{R}^a_m(A\setminus(A\setminus B))\\&=\mathfrak{R}^a_m\big((A\setminus(A\setminus B))\cup(B\setminus A)\big)\\&=\mathfrak{R}^a_m(B).\end{aligned}$$

(4) Assume that $A\in\mathscr{I}$. By Theorem 3.26(3), we get that $\mathfrak{R}^a_m(A)=X\setminus(X\setminus A)^{a^*}_m=X\setminus X^{a^*}_m$.

Theorem 3.33 Let $(X,m,\mathscr{I})$ be a minimal structure space with an ideal and $A\subseteq X$. Then $\mathfrak{R}^a_m(A)=\mathfrak{R}^a_m(\mathfrak{R}^a_m(A))$ ifandonlyif $(X\setminus A)^{a^*}_m=((X\setminus A)^{a^*}_m)^{a^*}_m$.

Proof It follows from the facts that,

I) $\mathfrak{R}^a_m(A)=X\setminus(X\setminus A)^{a^*}_m$ and $\mathfrak{R}_a$

II) $\mathfrak{R}^{a^*}_m(\mathfrak{R}^{a^*}_m(A))=X\setminus[X\setminus(X\setminus(X\setminus A)^{a^*}_m)]^{a^*}_m$ $=X\setminus((X\setminus A)^{a^*}_m)^{a^*}_m$.

Therefore $\mathfrak{R}^a_m(A)=\mathfrak{R}^a_m(\mathfrak{R}^a_m(A))$ if and only if $(X\setminus A)^{a^*}_m=((X\setminus A)^{a^*}_m)^{a^*}_m$.

Discussion and Conclusion

The aim of this article is to introduce the results of properties of some sets in a minimal structure space with an ideal. In addition, we study some properties of *δ-m-open* sets, *a-m-open* sets in a minimal structure space with an ideal. Moreover, we define an *δ-m-local* function and an $R^a{}_m$*-operator* in a minimal structure space with an ideal. Someproperties of them are obtained.

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