

ตัวกรองไฮเพอร์วิชันัยแบบอ่อนของพีชคณิตบีอีไฮเพอร์

Fuzzy Weak Hyper Filters of Hyper BE-Algebras

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บทคัดย่อ

ในบทความวิจัยนี้ได้แนะนำแนวคิดของตัวกรองไฮเพอร์วิชันัยแบบอ่อนในพีชคณิตบีอีไฮเพอร์ และได้ศึกษาสมบัติบางประการของตัวกรองไฮเพอร์วิชันัยแบบอ่อน จากนั้นได้แสดงว่าเซตของตัวกรองไฮเพอร์วิชันัยแบบอ่อนทั้งหมดของพีชคณิตบีอีไฮเพอร์เป็นแลตทิซบริบูรณ์ที่มีการแจกแจง ยิ่งไปกว่านั้นได้จำแนกลักษณะเฉพาะของพีชคณิตบีอีไฮเพอร์นอเทอร์เรียน และพีชคณิตบีอีไฮเพอร์อาร์ทีเนียน โดยใช้ตัวกรองไฮเพอร์วิชันัยแบบอ่อน

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Abstract

The aim of this work is to introduce the notion of fuzzy weak hyper filters in hyper BE-algebras and investigate some of their properties. This research shows that the set of all fuzzy weak hyper filters of hyper BE-algebras is a distributive complete lattice. Also, the concepts of Noetherian hyper BE-algebras and Artinian hyper BE-algebras are characterized by their fuzzy weak hyper filters.

Keywords: fuzzy hyper filter, fuzzy weak hyper filter, BE-algebra, hyper BE-algebra

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Introduction

The fuzzy set was introduced by Zadeh¹ as a function from a nonempty set X to the unit interval $[0,1]$. Later, many researchers have discussed the generalizations of the concepts of fuzzy sets with applications in computing, logic and many ramifications of pure and applied mathematics. Kim and Kim² introduced the notion of BE-algebras, as a generalization of BCK-algebras³ and BCI-algebras⁴. In 2010, the concept of fuzzy ideals in BE-algebras was introduced and some of its properties were investigated by Song, Jun and Lee⁵. Then, Dymek and Walendziak⁶ studied and characterized the concept of fuzzy filters in BE-algebras.

The hyperstructure theory was introduced by Marty⁷ in 1934 as a generalization of ordinary algebraic structures. Radfar, Rezaei and Borumand Saeid⁸ applied the hyper theory to introduce the notion of hyper BE-algebras, as a generalization of BE-algebras. In 2015, Cheng and Xin⁹ investigated some types of hyper filters on hyper BE-algebras.

In this work, the concept of fuzzy weak hyper filters of hyper BE-algebras is introduced, and its properties are considered. Finally, the concepts of Noetherian hyper BE-algebras and Artinian hyper BE-algebras are characterized by their fuzzy weak hyper filters.

Preliminaries

Let X be a nonempty set. The mapping $\circ, X \times X \rightarrow P^*(X)$, where $P^*(X)$ denotes the set of all nonempty subsets of H , is called a *hyperoperation*¹⁰⁻¹³ on H . The hyperstructure (H, \circ) is called a *hypergroupoid*. Let A and B be any two nonempty subsets of H and $x \in H$. Then, we denote

$$A \circ B = \bigcup_{a \in A, b \in B} a \circ b,$$

$$A \circ x = A \circ \{x\} \text{ and } x \circ B = \{x\} \circ B.$$

Let H be a nonempty set and $\circ: X \times X \rightarrow P^*(X)$ be a hyperoperation. Then (H, \circ, I) is called a *hyper BE-algebra*⁸ if it satisfies the following axioms:

- (i) $x < I$ and $x < x$;
- (ii) $x \circ (y \circ z) = y \circ (x \circ z)$;

(iii) $x \in I \circ x$;

(iv) $I < x$ implies $x = I$;

for all $x, y, z \in H$, where the relation “ $<$ ” is defined by $x < y$ if and only if $I \in x \circ y$.

Example 2.1⁸ Define the hyperoperation “ \circ ” on \mathbb{R} as follows:

$$x \circ y = \begin{cases} \{y\} & \text{if } x = 1; \\ \mathbb{R} & \text{otherwise.} \end{cases}$$

Then, (\mathbb{R}, \circ, I) is a hyper BE-algebra.

Example 2.2⁸ Let $X = \{1, a, b\}$. Define the hyperoperation “ \circ ” on as follows:

\circ	I	a	b
I	$\{I\}$	$\{a\}$	$\{b\}$
a	$\{I, a\}$	$\{I, a, b\}$	$\{I, a\}$
b	$\{I, a, b\}$	$\{a\}$	$\{I, a, b\}$

Then, (H, \circ, I) is a hyper BE-algebra.

Let F be a nonempty subset of a hyper BE-algebra H and $I \in F$. Then F is called:

- (i) a *weak hyper filter*⁸ of H if $x \circ y \subseteq F$ and $x \in F$, then $y \in F$, for all $x, y \in F$;
- (ii) a *hyper filter*⁸ of H if $x \circ y \approx F$ and $x \in F$, then $y \in F$, where $x \circ y \approx F$ means that $x \circ y \cap F \neq \emptyset$, for all $x, y \in H$.

Note that every hyper filter of a hyper BE-algebra H is a weak hyper filter of H , but the converse is not true in general⁸. In this paper, we will focus on weak hyper filters of hyper BE-algebras.

Lemma 2.3 If $\{F_i; i \in I\}$ is a chain of a family of weak hyper filters of a hyper BE-algebra H , then $\bigcup_{i \in I} F_i$ is also a weak hyper filter of H .

Proof. Let $\bigcup_{i \in I} F_i$. Clearly, $I \in F$. Let $x, y \in H$ such that $x \circ y \subseteq F$ and $x \in F$. Then $x \circ y \subseteq F_i$ and $x \in F_i$ for some $i, j \in I$. Assume that $F_i \subseteq F_j$. It follows that $x \circ y \subseteq F_j$ and $x \in F_j$. Since F_j is a weak hyper filter of H , we have $y \in F_j \subseteq F$. Hence, F is a weak hyper filter of H .

A *fuzzy set*¹ of a nonempty set X is a mapping $\mu: X \rightarrow [0,1]$. Then, the set $U(\mu; \alpha) = \{x \in X: \mu(x) \geq \alpha\}$ is called a *level subset* of μ . where $\alpha \in [0,1]$. Let μ and ν be any two fuzzy sets of a nonempty set X . Then $\mu \subseteq \nu$, means

that $\mu(x) \leq \nu(x)$, for all $x \in X$. In addition, the intersection and the union of μ and ν , denoted by $\mu \cap \nu$ and $\mu \cup \nu$, respectively, are defined by letting $x \in X$, $(\mu \cap \nu)(x) = \min\{\mu(x), \nu(x)\}$ and $(\mu \cup \nu)(x) = \max\{\mu(x), \nu(x)\}$.

Results

In this section, we introduce the notion of fuzzy weak hyper filters of hyper BE-algebras, and we investigate some fundamental properties of fuzzy weak hyper filters in hyper BE-algebras.

Definition 3.1 A fuzzy set μ of a hyper BE-algebra H is called a *fuzzy weak hyper filter* of H if it satisfies the following conditions:

- (i) $\mu(1) \geq \mu(x)$;
- (ii) $\mu(x) \geq \min\{\inf_{z \in y \circ x} \mu(z), \mu(y)\}$;

for all $x, y \in H$.

Example 3.2 Let $H = \{1, a, b\}$ be a set with a hyperoperation "o" on defined as follows:

o	1	a	b
1	{1}	{a, b}	{b}
a	{1}	{1, a}	{1, b}
b	{1}	{1, a, b}	{1}

Then, is a hyper BE-algebra⁸. We define a fuzzy set μ of H by $\mu(a) \leq \mu(b) \leq \mu(1)$. By routine computations, we have that μ is a fuzzy weak hyper filter of H .

Theorem 3.3 Let μ be a fuzzy set of a hyper BE-algebra H . Then μ is a fuzzy weak hyper filter of H if and only if its nonempty level subset $U(\mu; \alpha) = \{x \in H : \mu(x) \geq \alpha\}$ is a weak hyper filter of for all $\alpha \in [0, 1]$.

Proof. Assume that μ is a fuzzy weak hyper filter of H . Let $\alpha \in [0, 1]$ such that $U(\mu; \alpha) \neq \emptyset$. Then there exists $x_0 \in U(\mu; \alpha)$ such that $\mu(x_0) \geq \alpha$. Since $\mu(1) \geq \mu(x_0)$, $1 \in U(\mu; \alpha)$. Let $x, y \in H$ such that $x \circ y \subseteq U(\mu; \alpha)$ and $x \in U(\mu; \alpha)$. Then $\mu(z) \geq \alpha$, for all $z \in x \circ y$. Thus, $\mu(y) \geq \min\{\inf_{z \in x \circ y} \mu(z), \mu(x)\} \geq \alpha$, that is, $y \in U(\mu; \alpha)$. Hence, $U(\mu; \alpha)$ is a weak hyper filter of H .

Conversely, suppose that $\mu(1) \geq \mu(x_0) = \beta$ for some $x_0 \in H$ and $\beta \in [0, 1]$. Then $U(\mu; \beta) \neq \emptyset$, and so $U(\mu; \beta)$ is a weak hyper filter of H . It follows that $1 \in U(\mu; \beta)$, which implies that $\mu(1) \geq \beta$. This is a contradiction. Thus, $\mu(1) \geq \mu(x)$, for all $x \in H$. Suppose that $\mu(a) < \min\{\inf_{z \in b \circ a} \mu(z), \mu(b)\}$

for some $a, b \in H$. Letting $\alpha = \frac{1}{2}(\mu(a) + \min\{\inf_{z \in b \circ a} \mu(z), \mu(b)\})$.

We have $\mu(a) < \alpha < \min\{\inf_{z \in b \circ a} \mu(z), \mu(b)\} \leq \inf_{z \in b \circ a} \mu(z)$ and $\alpha < \mu(b)$. Then $b \circ a \subseteq U(\mu; \alpha)$ and $b \in U(\mu; \alpha)$. Since is a weak hyper filter of H , we have $a \in U(\mu; \alpha)$, that is, $\mu(a) \geq \alpha$. This is a contradiction. We obtain that $\mu(a) \geq \min\{\inf_{z \in b \circ a} \mu(z), \mu(b)\}$ for all $a, b \in H$. Therefore, μ is a fuzzy weak hyper filter of H .

Corollary 3.4 If μ is a fuzzy weak hyper filter of a hyper BE-algebra H , then the set $H_a = \{x \in H : \mu(x) \geq \mu(a)\}$ is a weak hyper filter of H for all $a \in H$.

Corollary 3.5 If μ is a fuzzy weak hyper filter of a hyper BE-algebra H , then the set $H_\mu = \{x \in H : \mu(x) = \mu(1)\}$ is a weak hyper filter of H .

Theorem 3.6 Let $F_1 \subset F_2 \subset \dots \subset F_n \subset \dots$ be a strictly ascending chain of weak hyper filters of a hyper BE-algebra H and $\{t_n\}$ be a strictly decreasing sequence in $[0, 1]$. Let μ be a fuzzy set of H , defined by $\mu(x) =$

$$\begin{cases} 0 & \text{if } x \notin F_n \quad \text{for each } n \in \mathbb{N}; \\ t_n & \text{if } x \in F_n - F_{n-1} \quad \text{for } n = 1, 2, \dots; \end{cases}$$

for all $x \in H$, where $F_0 = \emptyset$. Then μ is a fuzzy weak hyper filter of H .

Proof. Let $F = \bigcup_{n \in \mathbb{N}} F_n$. By Lemma 2.3, F is a weak hyper filter of H . Then $\mu(1) = t_1 \geq \mu(x)$, for all $x \in H$. Let $x, y \in H$. Thus, we can divide to be two cases, as follows.

Case 1: $x \notin F$. Then $y \circ x \notin F$ or $y \notin F$. There exists $a \in y \circ x$ such that $a \notin F$. Thus, $\mu(a) = 0$ or $\mu(y) = 0$. Hence, $\min\{\inf_{z \in y \circ x} \mu(z), \mu(y)\} = 0$.

Case 2: $x \in F_n - F_{n-1}$ for some $n = 1, 2, \dots$. Then $y \circ x \subseteq F_{n-1}$ or $y \notin F$. Thus, there exists $a \in y \circ x$ such that $a \in F_{n-1}$. We obtain that, $\inf_{z \in y \circ x} \mu(z) \leq t_{n-1}$ or $\mu(y) \leq t_{n-1}$. Therefore, $\min\{\inf_{z \in y \circ x} \mu(z), \mu(y)\} \leq t_n = \mu(x)$. Consequently, μ is a fuzzy weak hyper filter of H .

Let μ and ν be fuzzy sets of a nonempty set X . The *cartesian product*¹⁴ of μ and ν is defined by $(\mu \times \nu)(x, y) = \min\{\inf_{z \in y \circ x} \mu(z), \nu(y)\}$, for all $x, y \in X$.

Theorem 3.7 Let H be a hyper BE-algebra. If μ and ν are fuzzy weak hyper filters of H , then $\mu \times \nu$ is a fuzzy weak hyper filter of $H \times H$.

Proof. Assume that μ and ν are fuzzy weak hyper filters of H . Let $(x, y) \in H \times H$. Then

$$\begin{aligned}
 (\mu \times \nu)(1, 1) &= \min\{\mu(1), \nu(1)\} \geq \min\{\mu(x), \nu(y)\} \\
 &= (\mu \times \nu)(x, y). \text{ Now, let } (x_1, y_2), (x_2, y_2) \in H \times H. \text{ Then} \\
 &(\mu, \nu)(x_1, y_1) \\
 &= \min\{\mu(x_1), \nu(y_1)\} \\
 &\geq \min\{\min\{\inf_{z_1 \in x_2 \circ x_1} \mu(z_1), \mu(x_2)\}, \\
 &\quad \min\{\inf_{z_2 \in y_2 \circ y_1} \nu(z_2), \nu(y_2)\}\} \\
 &\geq \min\{\inf_{z_1 \in x_2 \circ x_1} \{\min\{\mu(z_1), \nu(z_2)\}, \\
 &\quad \inf_{z_2 \in y_2 \circ y_1} \{\min\{\mu(x_2), \nu(y_2)\}\}\} \\
 &\geq \min\{\inf_{(z_1, z_2) \in (x_2, y_2) \circ (x_1, y_1)} (\mu \times \nu)(z_1, z_2), \\
 &\quad (\mu \times \nu)(x_2, y_2)\}.
 \end{aligned}$$

Therefore, $\mu \times \nu$ is a fuzzy weak hyper filter of $H \times H$.

Let be a fuzzy set of a nonempty set X , $\alpha \in [0, 1 - \sup_{x \in X} \mu(x)]$ and $\beta \in [0, 1]$. Then:

- (i) the mapping $\mu_\alpha^T : X \rightarrow [0, 1]$ is called a *fuzzy translation*¹⁵ of μ if $\mu_\alpha^T(x) = \mu(x) + \alpha$, for all $x \in X$;
- (ii) the mapping $\mu_\beta^M : X \rightarrow [0, 1]$ is called a *fuzzy multiplication*¹⁵ of μ if $\mu_\beta^M(x) = \beta\mu(x)$, for all $x \in X$;
- (iii) the mapping $\mu_{\beta, \alpha}^{MT} : X \rightarrow [0, 1]$ is called a *fuzzy magnified translation*¹⁶ of μ if $\mu_{\beta, \alpha}^{MT}(x) = \beta\mu(x) + \alpha$, for all $x \in X$.

Theorem 3.8 Let H be a hyper BE-algebra, μ be a fuzzy set of H , $\alpha \in [0, 1 - \sup_{x \in H} \mu(x)]$ and $\beta \in [0, 1]$. Suppose that $\mu_{\beta, \alpha}^{MT}$ is a fuzzy magnified translation of μ , with respect to α and β . Then μ is a fuzzy weak hyper filter of H if and only if $\mu_{\beta, \alpha}^{MT}$ is a fuzzy weak hyper filter of H .

Proof. Assume that μ is a fuzzy weak hyper filter of H . Let $a \in H$. Since $\mu(1) \geq \mu(a)$, we have $\mu_{\beta, \alpha}^{MT}(1) = \beta\mu(1) + \alpha \geq \beta\mu(a) + \alpha = \mu_{\beta, \alpha}^{MT}(a)$, for all $a \in H$. Let $x, y \in H$. Then

$$\begin{aligned}
 \mu_{\beta, \alpha}^{MT}(x) &= \beta\mu(x) + \alpha \\
 &\geq \beta \min\{\inf_{z \in y \circ x} \mu(z), \mu(y)\} + \alpha \\
 &= \min\{\inf_{z \in y \circ x} (\beta\mu(z) + \alpha), \beta\mu(y) + \alpha\} \\
 &= \min\{\inf_{z \in y \circ x} \mu_{\beta, \alpha}^{MT}(z), \mu_{\beta, \alpha}^{MT}(y)\}.
 \end{aligned}$$

Hence, $\mu_{\beta, \alpha}^{MT}$ is a fuzzy weak hyper filter of H .

Conversely, assume that $\mu_{\beta, \alpha}^{MT}$ is a fuzzy weak hyper filter of H . Let $x, y \in H$. Consider $\beta\mu(1) + \alpha = \mu_{\beta, \alpha}^{MT}(1) \geq \mu_{\beta, \alpha}^{MT}(x) = \beta\mu(x) + \alpha$ and

$$\begin{aligned}
 \beta\mu(x) + \alpha &= \mu_{\beta, \alpha}^{MT}(x) \\
 &\geq \min\{\inf_{z \in y \circ x} \mu_{\beta, \alpha}^{MT}(z), \mu_{\beta, \alpha}^{MT}(y)\} \\
 &= \min\{\inf_{z \in y \circ x} (\beta\mu(z) + \alpha), \beta\mu(y) + \alpha\} \\
 &= \min\{\beta(\inf_{z \in y \circ x} \mu(z)) + \alpha, \beta\mu(y) + \alpha\} \\
 &= \beta \min\{\inf_{z \in y \circ x} \mu(z), \mu(y)\} + \alpha.
 \end{aligned}$$

Since $\beta > 0$ and $\alpha \geq 0$, we have $\mu(x) \geq \min\{\inf_{z \in y \circ x} \mu(z), \mu(y)\}$ and $\mu(1) \geq \mu(x)$, for all $x, y \in H$. Hence, μ is a fuzzy weak hyper filter of H .

Corollary 3.9 Let H be a hyper BE-algebra, μ be a fuzzy set of H , $\alpha \in [0, 1 - \sup_{x \in X} \mu(x)]$, and $\beta \in [0, 1]$. Suppose that μ_α^T is a fuzzy translation and is a fuzzy multiplication of with respect to and , respectively. Then the following conditions are equivalent:

- (i) μ is a fuzzy weak hyper filter of H ;
- (ii) μ_α^T is a fuzzy weak hyper filter of H ;
- (iii) μ_β^M is a fuzzy weak hyper filter of H .

Theorem 3.10 If μ and ν are fuzzy weak hyper filters of a hyper BE-algebra H , then $\mu \cap \nu$ is a fuzzy weak hyper filter of H .

Proof. Assume that μ and ν are fuzzy weak hyper filters of a hyper BE-algebra H . Let $x, y \in H$. Then

$$\begin{aligned}
 (\mu \cap \nu)(1) &= \min\{\mu(1), \nu(1)\} \\
 &\geq \min\{\mu(x), \nu(x)\} = (\mu \cap \nu)(x)
 \end{aligned}$$

and

$$\begin{aligned}
 (\mu \cap \nu)(x) &= \min\{\mu(x), \nu(x)\} \\
 &\geq \min\{\min\{\inf_{z \in y \circ x} \mu(z), \mu(y)\}, \\
 &\quad \min\{\inf_{z \in y \circ x} \nu(z), \nu(y)\}\} \\
 &= \min\{\inf_{z \in y \circ x} \{\min\{\mu(z), \nu(z)\}\}, \\
 &\quad \min\{\mu(y), \nu(y)\}\} \\
 &= \min\{\inf_{z \in y \circ x} (\mu \cap \nu)(z), (\mu \cap \nu)(y)\}.
 \end{aligned}$$

Hence, $\mu \cap \nu$ is a fuzzy weak hyper filter of H .

Theorem 3.11 If μ and ν are fuzzy weak hyper filters of a hyper BE-algebra H such that $\mu \subseteq \nu$ or $\nu \subseteq \mu$, then $\mu \cup \nu$ is a fuzzy weak hyper filter of H .

Proof. Assume that μ and ν are fuzzy weak hyper filters of a hyper BE-algebra H such that $\mu \subseteq \nu$ or $\nu \subseteq \mu$. Let $x, y \in H$. Then

$$\begin{aligned}
 (\mu \cup \nu)(1) &= \max\{\mu(1), \nu(1)\} \\
 &\geq \max\{\mu(x), \nu(x)\} = (\mu \cup \nu)(x).
 \end{aligned}$$

Now,

$$\begin{aligned}
 (\mu \cup \nu)(x) &= \max\{\mu(x), \nu(x)\} \\
 &\geq \max\{\min\{\inf_{z \in y \circ x} \mu(z), \mu(y)\}, \\
 &\quad \min\{\inf_{z \in y \circ x} \nu(z), \nu(y)\}\} \\
 &= \min\{\max\{\inf_{z \in y \circ x} \mu(z), \mu(y)\}, \\
 &\quad \max\{\inf_{z \in y \circ x} \nu(z), \nu(y)\}\} \\
 &= \min\{\inf_{z \in y \circ x} \{\max\{\mu(z), \nu(z)\}\}, \\
 &\quad \max\{\mu(y), \nu(y)\}\} \\
 &= \min\{\inf_{z \in y \circ x} (\mu \cup \nu)(z), (\mu \cup \nu)(y)\}.
 \end{aligned}$$

In general, $\max\{\min\{\}\}\min\{\max\{\}\}$. Suppose for this case

$$\begin{aligned}
 &\max\{\min\{\inf_{z \in y \circ x} \mu(z), \mu(y)\}, \\
 &\quad \min\{\inf_{z \in y \circ x} \nu(z), \nu(y)\}\} \\
 &\neq \min\{\max\{\inf_{z \in y \circ x} \mu(z), \mu(y)\}, \\
 &\quad \max\{\inf_{z \in y \circ x} \nu(z), \nu(y)\}\}.
 \end{aligned}$$

Then there exists $\alpha \in [0,1]$ such that

$$\begin{aligned}
 &\max\{\min\{\inf_{z \in y \circ x} \mu(z), \mu(y)\}, \\
 &\quad \min\{\inf_{z \in y \circ x} \nu(z), \nu(y)\}\} \\
 &< \alpha < \min\{\max\{\inf_{z \in y \circ x} \mu(z), \mu(y)\}, \\
 &\quad \max\{\inf_{z \in y \circ x} \nu(z), \nu(y)\}\}.
 \end{aligned}$$

Thus, $\alpha < \min\{\inf_{z \in y \circ x} \mu(z), \mu(y)\}$. On the other hand, $\min\{\inf_{z \in y \circ x} \mu(z), \mu(y)\} < \alpha$, which is a contradiction. This completes the proof.

Then, we have the following corollary.

Corollary 3.12 Let $\{\mu_i, i \in \Lambda\}$ be a nonempty set of a family of fuzzy weak hyper filters of a hyper BE-algebra H , where Λ is an arbitrary indexed set. Then the following statements hold:

- (i) $\bigcap_{i \in \Lambda} \mu_i$ is a fuzzy weak hyper filter of H ;
- (ii) if $\mu_i \subseteq \mu_j$ or $\mu_j \subseteq \mu_i$ for all $i, j \in \Lambda$, then $\bigcap_{i \in \Lambda} \mu_i$ is a fuzzy weak hyper filter of H .

Next, we denote by $FHF(H)$ the set of all fuzzy weak hyper filters of a hyper BE-algebra H . By Corollary 3.12, we obtain the following theorem.

Theorem 3.13 Let H be a hyper BE-algebra and $(FHF(H); \subseteq)$ be a totally ordered set by the set inclusion. Then $(FHF(H); \subseteq, \vee, \wedge)$ is a complete lattice, where

$$\begin{aligned}
 \wedge\{\mu_i \in FHF(H) : i \in \Lambda\} &= \bigcap_{i \in \Lambda} \mu_i, \\
 \vee\{\mu_i \in FHF(H) : i \in \Lambda\} &= \bigcap_{i \in \Lambda} \mu_i.
 \end{aligned}$$

Lemma 3.14 Let H be a hyper BE-algebra and $(FHF(H); \subseteq)$ be a totally ordered set. Then $\mu \cap (\nu \cup \lambda) = (\mu \cap \nu) \cup (\mu \cap \lambda)$ and $\mu \cup (\nu \cap \lambda) = (\mu \cup \nu) \cap (\mu \cup \lambda)$, for all $\mu, \nu, \lambda \in FHF(H)$.

Proof. Let $\mu, \nu, \lambda \in FHF(H)$ and $x \in H$. Then $(\mu \cap (\nu \cup \lambda))(x)$

$$\begin{aligned}
 &= \min\{\mu(x), (\nu \cup \lambda)(x)\} \\
 &= \min\{\mu(x), \max\{\nu(x), \lambda(x)\}\} \\
 &= \max\{\min\{\mu(x), \nu(x)\}, \min\{\mu(x), \lambda(x)\}\} \\
 &= \max\{(\mu \cap \nu)(x), (\mu \cap \lambda)(x)\} \\
 &= ((\mu \cap \nu) \cup (\mu \cap \lambda))(x).
 \end{aligned}$$

Hence, $\mu \cap (\nu \cup \lambda) = (\mu \cap \nu) \cup (\mu \cap \lambda)$. Similarly, we can prove that $\mu \cup (\nu \cap \lambda) = (\mu \cup \nu) \cap (\mu \cup \lambda)$.

From Lemma 3.14, we have the following theorem.

Theorem 3.15 Let be a hyper BE-algebra and be a totally ordered set. Then is a distributive complete lattice.

Next, we characterize Noetherian hyper BE-algebras and Artinian hyper BE-algebras using their fuzzy weak hyper filters.

A hyper BE-algebra H is called *Noetherian* if H satisfies the ascending chain condition on weak hyper filters, that is, for any weak hyper filters F_1, F_2, F_3, \dots of H , with $F_1 \subseteq F_2 \subseteq F_3 \subseteq \dots \subseteq F_i \subseteq \dots$

There exists $n \in \mathbb{N}$ such that $F_i = F_i + I$ for all $i \geq n$.

A hyper BE-algebra H is called *Artinian* if H satisfies the descending chain condition on weak hyper filters, that is, for any weak hyper filters F_1, F_2, F_3, \dots of H , with $F_1 \subseteq F_2 \subseteq F_3 \subseteq \dots \subseteq F_i \subseteq \dots$

There exists $n \in \mathbb{N}$ such that $F_i = F_i + I$ for all $i \geq n$.

Theorem 3.16 Let H be a hyper BE-algebra. Then H is Noetherian if and only if for every fuzzy weak

hyper filter μ of H , the set $Im(\mu) = \{\mu(x):x \in H\}$ is a well-ordered subset of $[0,1]$.

Proof. Assume that H is Noetherian. Suppose that there exists a fuzzy weak hyper filter μ of H such that $Im(\mu)$ is not a well-ordered subset of $[0,1]$. Then there exists a strictly infinite decreasing sequence $\{t_n\}_{n=1}^\infty$ such that $\mu(x_n) = t_n$ for some $x_n \in H$. Let $I_n = U(\mu; t_n) = \{x \in H: \mu(x) \geq t_n\}$. By Theorem 3.3, I_n is a weak hyper filter of H , for all $n \in \mathbb{N}$. Moreover, $I_1 \subset I_2 \subset I_3 \subset \dots$ is a strictly infinite ascending chain of weak hyper filters of H . This is a contradiction that H is Noetherian. Therefore, $Im(\mu)$ is a well-ordered subset of $[0,1]$, for each fuzzy weak hyper filter μ of H .

Conversely, assume that for every fuzzy weak hyper filter μ of H , the set $Im(\mu) = \{\mu(x): x \in H\}$ is a well-ordered subset of $[0,1]$. Suppose that H is not Noetherian. Then there exists a strictly infinite ascending chain $F_1 \subset F_2 \subset F_3 \subset \dots \subset F_n \subset \dots$ of weak hyper filters of H . We define the fuzzy weak hyper filter of μ of H by

$$\mu(x) = \begin{cases} 0 & \text{if } x \notin F_n \text{ for each } n \in \mathbb{N}; \\ \frac{1}{n} & \text{if } x \in F_n - F_{n-1} \text{ for } n = 1, 2, \dots; \end{cases}$$

where $F_0 = \emptyset$. By Theorem 3.6, μ is a fuzzy weak hyper filter of H , but $Im(\mu)$ is not a well-ordered subset of $[0,1]$. We get a contradiction. Consequently, H is Noetherian.

Corollary 3.17 Let H be a hyper BE-algebra. If for every fuzzy weak hyper filter μ of H such that $Im(\mu)$ is a finite set, then H is Noetherian.

Theorem 3.18 Let H be a hyper BE-algebra and $T = \{t_1, t_2, \dots\} \cup \{0\}$, where $\{t_n\}_{n=1}^\infty$ is a strictly decreasing in $[0,1]$. Then the following conditions are equivalent:

- (i) H is Noetherian;
- (ii) for every fuzzy weak hyper filter μ of H ,

if $Im(\mu) \subseteq T$, then there exists $k \in \mathbb{N}$ such that $Im(\mu) \subseteq \{t_1, t_2, \dots, t_k\} \cup \{0\}$.

Proof. (i) \Rightarrow (ii): Assume that H is a Noetherian. Let μ be a fuzzy weak hyper filter of H such that $Im(\mu) \subseteq T$. By Theorem 3.16, $Im(\mu)$ is a well-ordered subset of $[0,1]$. Hence, there exists $k \in \mathbb{N}$ such that $Im(\mu) \subseteq \{t_1, t_2, \dots, t_k\} \cup \{0\}$.

(ii) \Rightarrow (i): Assume that for every fuzzy weak hyper filter μ of H , if $Im(\mu) \subseteq T$, then there exists $k \in \mathbb{N}$ such that $Im(\mu) \subseteq \{t_1, t_2, \dots, t_k\} \cup \{0\}$. Suppose that H is not Noetherian. Then there exists a strictly ascending chain $F_1 \subset F_2 \subset F_3 \subset \dots$ of weak hyper filters of H . We define a fuzzy set μ of H by

$$\mu(x) = \begin{cases} 0 & \text{if } x \notin F_n \text{ for each } n \in \mathbb{N}; \\ t_n & \text{if } x \in F_n - F_{n-1} \text{ for } n = 1, 2, \dots; \end{cases}$$

where $F_0 = \emptyset$. By Theorem 3.6, μ is a fuzzy weak hyper filter of H . This is a contradiction with our assumption. Therefore, H is Noetherian.

Theorem 3.19 Let H be a hyper BE-algebra and $T = \{t_1, t_2, \dots\} \cup \{0\}$, where $\{t_n\}_{n=1}^\infty$ is a strictly increasing sequence in $[0,1]$. Then the following conditions are equivalent:

- (i) H is Artinian;
- (ii) for every fuzzy weak hyper filter μ of H ,

if $Im(\mu) \subseteq T$, then there exists $k \in \mathbb{N}$ such that $Im(\mu) \subseteq \{t_1, t_2, \dots, t_k\} \cup \{0\}$.

Proof. (i) \Rightarrow (ii): Assume that H is Artinian. Let μ be a fuzzy weak hyper filter of H such that $Im(\mu) \subseteq T$. Suppose that $t_{i_1} < t_{i_2} < \dots < t_{i_m} < \dots$ is a strictly increasing sequence of elements in $Im(\mu)$. Let $I_m = U(\mu; t_{i_m})$ for $m=1,2,\dots$. This implies that $I_1 \supset I_2 \supset \dots \supset I_m \supset \dots$ is a strictly descending chain of weak hyper filters μ of H , which is a contradiction that H is Artinian.

(ii) \Rightarrow (i): Assume that for every fuzzy weak hyper filter μ of H , if $Im(\mu) \subseteq T$, then there exists $k \in \mathbb{N}$ such that $Im(\mu) \subseteq \{t_1, t_2, \dots, t_k\} \cup \{0\}$. Suppose that H is not Artinian. Then there exists a strictly descending chain $F_1 \supset F_2 \supset \dots \supset F_n \supset \dots$ of weak hyper filters of H . We define a fuzzy set μ in H by

$$\mu(x) = \begin{cases} 0 & \text{if } x \notin F_1, \\ t_n & \text{if } x \in F_n - F_{n+1} \text{ for } n = 1, 2, \dots, \\ 1 & \text{if } x \in F_n \text{ for all } n \in \mathbb{N}. \end{cases}$$

We have that $\mu(I) = 1 \geq \mu(x)$, for all $x \in H$. Next, let $x, y \in H$. Thus, we can divide to be three cases, as follows.

Case 1: $x \notin F_1$. Then $y \circ x \notin F_1$ or $y \notin F_1$. Thus,

Case 2: $x \in F_n - F_{n+1}$ for some $n=1,2,\dots$. Then $y \circ x \notin F_{n+1}$ or $y \notin F_{n+1}$. We obtain that $\mu(y) \leq t_n$ or $\mu(z) \leq t_n$ for some $z \in y \circ x \setminus F_{n+1}$. So, $\min\{\inf_{z \in y \circ x} \mu(z), \mu(y)\} \leq t_n = \mu(x)$.

Case 3: $x \in F_n$ for all $n \in \mathbb{N}$. Clearly, $\mu(x) = 1 \geq \min\{\inf_{z \in y \circ x} \mu(z), \mu(y)\}$.

Hence, μ is a fuzzy weak hyper filter of H . We have a contradiction with our assumption. Consequently, H is Artinian.

Corollary 3.20 Let H be a hyper BE-algebra. If for every fuzzy weak hyper filter μ of H , $\text{Im}(\mu)$ is a finite set, then H is Artinian.

Conclusions

The concept of fuzzy weak hyper filters in hyper BE-algebras is introduced and investigated. It was shown that the set of all fuzzy weak hyper filters of hyper BE-algebras is a distributive complete lattice. Also, the concepts of Noetherian hyper BE-algebras and Artinian hyper BE-algebras are characterized by their fuzzy weak hyper filters. In future work, we will study the concept of characterizations of fuzzy weak hyper filters in hyper BE-algebras.

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