

ความสัมพันธ์ของกรีนบนโมนอยด์ของโคไฮเพอร์ซัพสตีติวชันเชิงเส้นชนิด $\tau = (n)$ Green's Relations on the Monoid of Linear Cohypersubstitutions of Type $\tau = (n)$

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บทคัดย่อ

โคไฮเพอร์ซัพสตีติวชันเชิงเส้นชนิด $\tau = (n)$ เป็นการส่งสัญลักษณ์การดำเนินการร่วมแบบ n -ary ไปยังพจน์ร่วมเชิงเส้นชนิด τ . สำหรับทุกโคไฮเพอร์ซัพสตีติวชันเชิงเส้น σ ชนิด $\tau = (n)$ ทำให้เกิดการส่ง $\hat{\sigma}$ บนเซตของพจน์ร่วมเชิงเส้นชนิด τ . ทั้งหมดเซตของโคไฮเพอร์ซัพสตีติวชันเชิงเส้นชนิด τ ทั้งหมด ภายใต้การดำเนินการทวิภาค \circ_{coh} ซึ่งถูกกำหนดนิยามโดย $\sigma_1 \circ_{coh} \sigma_2 := \hat{\sigma}_1 \circ \sigma_2$ สำหรับทุก $\sigma_1, \sigma_2 \in Cohyp^{lin}(n)$ เป็นโมนอยด์ ในนี้เรารู้จักแ่งลักษณะความสัมพันธ์ของกรีนบน $Cohyp^{lin}(n)$.

คำสำคัญ: โคไฮเพอร์ซัพสตีติวชันเชิงเส้น พจน์ร่วมเชิงเส้น การซ้อนทับ ความสัมพันธ์ของกรีน

Abstract

Linear cohypersubstitutions of type $\tau = (n)$ are mappings which map the n -ary co-operation symbols to linear coterms of type τ . Every linear cohypersubstitution σ of type $\tau = (n)$ induces a mapping $\hat{\sigma}$ on the set of all linear coterms of type τ . The set of all linear cohypersubstitutions of type τ under the binary operation \circ_{coh} which is defined by $\sigma_1 \circ_{coh} \sigma_2 := \hat{\sigma}_1 \circ \sigma_2$ for all $\sigma_1, \sigma_2 \in Cohyp^{lin}(n)$ forms a monoid. In this paper, we characterize Green's relations on $Cohyp^{lin}(n)$.

Keywords: linear cohypersubstitutions, linear coterms, superposition, Green's relations.

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Introduction

Let A be a non-empty set and n be a positive integer. The n -th copower $A^{\cup n}$ is the Cartesian product $A^{\cup n} := n \times A$, where $\underline{n} := 1, \dots, n$. An element (i, a) in the copower corresponds to the element a in the i -th copy of A , for $1 \leq i \leq n$. A co-operation on A is a mapping $f^A: A \rightarrow A^{\cup n}$ for some $n \geq 1$; the natural number n is called the arity of the co-operation f^A . We also need to recall that any n -ary co-operation f^A on set A can be uniquely expressed as a pair (f_1^A, f_2^A) of mappings, $f_1^A: A \rightarrow \underline{n}$ and $f_2^A: A \rightarrow A$; the first mapping gives the labeling used by f^A in mapping elements to copies of A , and the second mapping tells us what element of A is mapped to.

We shall denote by $cO_A^{(n)} = \{f^A \mid A \rightarrow A^{\cup n}\}$ the set of all n -ary co-operations defined on A , and by $cO_A := \cup_{n \geq 1} cO_A^{(n)}$ the set of all finitary co-operations defined on A . An indexed coalgebra is a pair $(A; (f_i^A)_{i \in I})$, where f_i^A is a n_i -ary co-operation defined on A , and $\tau = (n_i)_{i \in I}$ for $n_i \geq 1$ is called the type of the coalgebra. Coalgebras were studied by Drbohlav¹. In², the following superposition of co-operations was introduced. If $f^A \in cO_A^{(n)}$ and $g_0^A, \dots, g_{n-1}^A \in cO_A^{(k)}$ then the k -ary co-operation $f^A[g_0^A, \dots, g_{n-1}^A]: A \rightarrow A^{\cup k}$ is defined by $a \mapsto ((g_{f_1^A(a)}^A)_1(f_2^A(a)), (g_{f_2^A(a)}^A)_2(f_2^A(a)))$ for all $a \in A$. The co-operation $f^A[g_0^A, \dots, g_{n-1}^A]$ is called the superposition of f^A and g_0^A, \dots, g_{n-1}^A . It will also be denoted by $comp_k^n(f^A, g_0^A, \dots, g_{n-1}^A)$.

The injection co-operations $i_i^{n,A}: A \rightarrow A^{\cup n}$ are special co-operations which are defined for each $0 \leq i \leq n-1$ by $i_i^{n,A}: A \rightarrow A^{\cup n}$ with $a \mapsto (i, a)$ for all $a \in A$. Then we get a multi-based algebra $((cO_A^{(n)})_{n \geq 1}, (comp_k^n)_{k, n \geq 1}, (i_i^{n,A})_{0 \leq i \leq n-1})$, called the clone of co-operations on A . In², it is mentioned that this algebra is a clone, i.e. it satisfies the three clone axioms. In³, K. Denecke and K. Saengsura gave a full proof of this fact and introduced the following coterminals of type $\tau = (n_i)_{i \in I}$ were introduced. Let $(f_i)_{i \in I}$ be an indexed set of co-operation symbols such that for each $i \in I$. We say that symbol f_i has arity n_i , for $i \in I$. Let $U\{e_i^n \mid n \geq 1, n \in N, 0 \leq j \leq n-1\}$ be a set of symbols which is disjoint from the set $\{f_i \mid i \in I\}$. We assign to each e_j^n the positive integer n as its arity. Then coterminals of type τ are defined as follows:

- (i) For every $i \in I$, the co-operation symbol f_i is an n_i -ary coterminal of type τ .
- (ii) For every $n \geq 1$ and $0 \leq j \leq n-1$, the symbol e_j^n is an n -ary coterminal of type τ .
- (iii) If t_1, \dots, t_{n_i} are n_i -ary coterminals of type τ , then $f_i[t_1, \dots, t_{n_i}]$ is an n_i -ary coterminal of type τ and if t_0, \dots, t_{n-1} are m -ary coterminals of type τ , then $e_j^n[t_0, \dots, t_{n-1}]$ is an m -ary coterminal of type τ , for every $i \in I$ and $n \geq 1$ and $0 \leq j \leq n-1$.

Let $cT_i^{(n)}$ be the set of all n -ary coterminals of type τ and let $cT_i := \bigcup_{n \geq 1} cT_i^{(n)}$ be the set of all (finitary) coterminals of type τ .

Definition 1.1 Let $t \in cT_i$ be a coterminal and $E(t) = \{e_i^n \mid e_i^n \text{ occurs in } t \text{ and } 0 \leq i \leq n-1\}$. Then t is a linear coterminal if for each $e_i^n \in E(t)$, e_i^n occurs only once in t .

We denote by $cT_i^{lin,(n)}$ the set of all n -ary linear coterminals of type τ and $cT_i^{lin} := \bigcup_{n \geq 1} cT_i^{lin,(n)}$ the set of all (finitary) linear coterminals of type τ .

We define a family of superposition operations $(\bar{S}_m^n)_{m, n \geq 1}$ on this sequence, as follows.

Definition 1.2 The operation $\bar{S}_m^n: cT_i^{lin,(n)} \times (cT_i^{lin,(m)})^n \rightarrow cT_i^{lin,(m)}$ is defined by induction on the complexity of linear coterminal definition, as follows:

- (i) If e_i^n is an n -ary linear coterminal of type τ , t_0, \dots, t_{n-1} are m -ary linear coterminals of type τ for $0 \leq j \leq n-1$ and $E(t_j) \cap E(t_k) = \emptyset$ for $j, k \in \{0, \dots, n-1\}$ and $j \neq k$, then $\bar{S}_m^n(e_i^n, t_0, \dots, t_{n-1}) := t_i$ is an m -ary linear coterminal of type τ .
- (ii) If f is an n -ary linear coterminal of type τ , t_1, \dots, t_n are m -ary linear coterminals of type τ and $E(t_j) \cap E(t_k) = \emptyset$ for $j, k \in \{1, \dots, n\}$, then $\bar{S}_m^n(f, t_1, \dots, t_n) := f[t_1, \dots, t_n]$ is an n -ary linear coterminal of type τ .
- (iii) If f is an n -ary co-operation symbol, S_1, \dots, S_n are n -ary linear coterminals of type τ where $E(S_j) \cap E(S_k) = \emptyset$ for $j, k \in \{1, \dots, n\}$ and t_1, \dots, t_n are m -ary linear coterminals of type τ where $E(t_j) \cap E(t_k) = \emptyset$ for $j, k \in \{1, \dots, n\}$, then $\bar{S}_m^n(f[S_1, \dots, S_n], t_1, \dots, t_n) := f[\bar{S}_m^n(S_1, t_1, \dots, t_n), \dots, \bar{S}_m^n(S_n, t_1, \dots, t_n)]$ is an n -ary linear coterminal of type t .

Together with these operations we obtain a heterogeneous algebra $cT_t^{lin} := ((cT_t^{lin,(n)})_{n \geq 1}, (\bar{S}_m^n)_{m,n \geq 1}, (e_j^n)_{0 \leq j \leq n-1})$.

Definition 1.3 A linear cohypersubstitution of type t is a mapping $S : \{f\} \rightarrow cT_t^{lin}$ from the set of all co-operation symbols to the set of all linear coterms which is inductively defined by the following steps:

- (i) $\hat{\sigma}[e_j^n] := e_j^n$ for every $n \geq 1$ and $0 \leq j \leq n-1$,
- (ii) $\hat{\sigma}[f] := \sigma[f]$,
- (iii) $\hat{\sigma}[f[t_1, \dots, t_n]] := \bar{S}_n^n(\sigma(f), \hat{\sigma}[t_1], \dots, \hat{\sigma}[t_n])$ and assume that $\hat{\sigma}[t_j]$ is already defined and $E(t_j)$ are distinct for all $1 \leq j \leq n$.

Let $Cohyp^{lin}(\tau)$ be the set of all linear cohypersubstitutions of type τ . Since the extension of a linear cohypersubstitution of type τ maps cT_τ^{lin} to cT_τ^{lin} , we may define a binary operation o_{coh} by $\hat{\sigma}_1 o_{coh} \sigma_2 := \hat{\sigma}_1 o \sigma_2$ where o is the usual composition of mappings. Let σ_{id} be the linear cohypersubstitution defined by $\sigma_{id}(f) := f$.

In 2016, D. Boonchari and K. Saengsura studied the monoid of cohypersubstitutions of type $\tau = (n)^4$. In this paper, we characterize Green's relations on $Cohyp^{lin}(n)$.

Main results

In this section, we obtain the linear cohypersubstitutions σ_t and σ_s which are R -related, L -related, H -related, D -related and J -related as following theorem:

We characterize the Green's relation R on $Cohyp^{lin}(n)$ and we recall the definition of Green's relation R i.e., let a, b be elements of semigroup S . Then $a R b$ if and only if there exists x, y in S such that $xa = b, yb = a$.

Theorem 2.1 Let $\sigma_t, \sigma_s \in Cohyp^{lin}(n)$. If $t = e_i^n, s = e_j^n \in cT_t^{lin,(n)}$ for all $i, j \in \{0, \dots, n-1\}$ then $\sigma_t R \sigma_s$.

Proof Assume that $t = e_i^n, s = e_j^n \in cT_t^{lin,(n)}$ for all $i, j \in \{0, \dots, n-1\}$. We will show that there are $\sigma_r, \sigma_w \in Cohyp^{lin}(n)$ such that $\sigma_t = \sigma_s o_{coh} \sigma_r$ and $\sigma_s = \sigma_t o_{coh} \sigma_w$.

$$\begin{aligned} \text{Since } \sigma_s(f) = s = e_j^n \text{ and } \hat{\sigma}_t[e_j^n] = e_j^n, \text{ then } \sigma_s(f) &= e_j^n \\ &= \hat{\sigma}_t[e_j^n] \\ &= \hat{\sigma}_t[\sigma_{e_j^n}(f)] \end{aligned}$$

$$\begin{aligned} &= \hat{\sigma}_t[\sigma_s(f)] \\ &= (\sigma_t o_{coh} \sigma_s)(f). \end{aligned}$$

Therefore, $\sigma_s = \sigma_t o_{coh} \sigma_r$.

Similarly, one can show that $\sigma_t = \sigma_s o_{coh} \sigma_r$ for some $\sigma_r \in Cohyp^{lin}(n)$.

This implies that $\sigma_t R \sigma_s$.

Theorem 2.2 Let $\sigma_t, \sigma_s \in Cohyp^{lin}(n)$. If $t = f[e_{j_0}^n, \dots, e_{j_{n-1}}^n] \in cT_t^{lin,(n)}$ and $s = f[e_{i_0}^n, \dots, e_{i_{n-1}}^n] \in cT_s^{lin,(n)}$ where $i_0, \dots, i_{n-1}, j_0, \dots, j_{n-1} \in \{0, \dots, n-1\}$ then $\sigma_t R \sigma_s$.

Proof Let $r = f[r_1, \dots, r_n] \in cT_t^{lin,(n)}$ such that $r_{j_k} = e_{i_k}^n$ for all $j_k \in \{0, \dots, n-1\}$ and $k = 0, \dots, n-1$.

Then $\sigma_t(f) = f[e_{j_0}^n, \dots, e_{j_{n-1}}^n]$, and so $(\sigma_s o_{coh} \sigma_r)(f) = [\hat{\sigma}_s(f)]$

$$\begin{aligned} &= \hat{\sigma}_s[f[r_1, \dots, r_n]] \\ &= \sigma_s(f)[r_1, \dots, r_n] \\ &= f[e_{j_0}^n, \dots, e_{j_{n-1}}^n][r_1, \dots, r_n] \\ &= f[e_{j_0}^n[r_1, \dots, r_n], \dots, e_{j_{n-1}}^n[r_1, \dots, r_n]] \\ &= f[e_{i_0}^n, \dots, e_{i_{n-1}}^n] \\ &= t \\ &= \sigma_t(f). \end{aligned}$$

Therefore, $\sigma_s o_{coh} \sigma_r = \sigma_t$.

Similarly, one can show that $\sigma_s = \sigma_t o_{coh} \sigma_w$ for some $\sigma_w \in Cohyp^{lin}(n)$.

Hence, $\sigma_t R \sigma_s$.

Therefore, $(\sigma_t, \sigma_s) \in R$.

For linear cohypersubstitutions σ_t, σ_s such that t and s are different form i.e., $t \in \{e_i^n \mid 0 \leq i \leq n-1\}$ and $s \in cT_t^{lin,(n)} \setminus \{e_i^n \mid 0 \leq i \leq n-1\}$, we have that $(\sigma_t, \sigma_s) \notin R$ as the following example:

Example 2.3 Let $\sigma_t, \sigma_s \in Cohyp^{lin}(n)$ and $t = e_i^n, s = f[e_{j_0}^n, \dots, e_{j_{n-1}}^n] \in cT_t^{lin,(n)}$ for all $i, j_0, \dots, j_{n-1} \in \{0, \dots, n-1\}$ and $E(s)$ be distinct.

Assume that $(\sigma_t, \sigma_s) \in R$.

Then there is $\sigma_w \in Cohyp^{lin}(n)$. such that $\sigma_s = \sigma_t o_{coh} \sigma_w$.

Hence

$$f[e_{j_0}^n, \dots, e_{j_{n-1}}^n] = s$$

$$\begin{aligned} &= \sigma_s(f) \\ &= \hat{\sigma}_t[\sigma_w(f)] \\ &= \hat{\sigma}_t[w]. \end{aligned}$$

But we cannot find $w \in cT_t^{lin,(n)}$ such that

$$\hat{\sigma}_t[w] = ff[e_{j_1}^n, \dots, e_{j_n}^n].$$

So $(\sigma_t, \sigma_s) \in R$.

Remark The number of pairs (σ_t, σ_s) in which $\sigma_t R \sigma_s$ is $n^2 + (n!)^2$.

Next, we characterize the Green's relation L on $Cohyp^{lin}(n)$ and we recall the definition of Green's relation L i.e., $a L b$ if and only if there exists u, v in S such that $au = v, bv = u$.

Theorem 2.4 Let $\sigma_t, \sigma_s \in Cohyp^{lin}(n)$ and $t, s \in \{e_i^n \mid n \geq 1, 0 \leq i \leq n-1\}$. If $\sigma_t L \sigma_s$, then $t = s$.

Proof Assume that $\sigma_t L \sigma_s$.

Then there are $\sigma_u, \sigma_v \in Cohyp^{lin}(n)$ such that $\sigma_t = \sigma_u o_{coh} \sigma_s$ and $\sigma_s = \sigma_v o_{coh} \sigma_t$.

Let $\sigma_t(f) = t = e_j^n$ and $\sigma_s(f) = s = e_j^n$.

Then

$$\begin{aligned} e_i^n &= t \\ &= \sigma_t(f) \\ &= \hat{\sigma}_u[\sigma_s(f)] \\ &= \hat{\sigma}_u[e_i^n] \\ &= e_i^n \\ &= s. \end{aligned}$$

Therefore, $t = s$.

For linear cohypersubstitutions σ_t, σ_s such that $t, s \in \{e_i^n \mid 0 \leq i \leq n-1\}$. and $t \neq s$, we have that $(\sigma_t, \sigma_s) \in L$ as the following example:

Example 2.5 Let $\sigma_t, \sigma_s \in Cohyp^{lin}(n)$

Assume that $t = e_i^n, s = e_j^n \in cT_t^{lin,(n)}$ for all $i, j \in \{0, \dots, n-1\}$ and $i \neq j$.

Then $e_i^n = t = \sigma_t(f)$ and $e_j^n = s = \sigma_s(f)$.

Since for all $\sigma_s \in Cohyp^{lin}(n)$, we have that $\hat{\sigma}_u[e_j^n] = e_j^n$. Then $\hat{\sigma}_u[\sigma_s(f)] = \sigma_s(f) \neq \sigma_t(f)$.

Therefore, $(\sigma_t, \sigma_s) \in L$.

Theorem 2.6 If $t = ff[e_{i_0}^n, \dots, e_{i_{n-1}}^n] \in cT_t^{lin,(n)}$ and $s = ff[e_{j_0}^n, \dots, e_{j_{n-1}}^n] \in cT_s^{lin,(n)}$ where $i_0, \dots, i_{n-1}, j_0, \dots, j_{n-1} \in \{0, \dots, n-1\}$, then $\sigma_t L \sigma_s$.

Proof Let $v = ff[v_1, \dots, v_n] \in cT_t^{lin,(n)}$ such that $v_1, \dots, v_n \in \{e_i^n \mid i = 0, \dots, n-1\}$ and $v_1[e_{i_0}^n, \dots, e_{i_{n-1}}^n] = e_{j_0}^n, \dots, e_n[e_{i_0}^n, \dots, e_{i_{n-1}}^n] = e_{j_{n-1}}^n$.

Then

$$\begin{aligned} \hat{\sigma}_v[\sigma_t(f)] &= \hat{\sigma}_v[ff[e_{i_0}^n, \dots, e_{i_{n-1}}^n]] \\ &= \sigma_v(f)[e_{i_0}^n, \dots, e_{i_{n-1}}^n] \\ &= (ff[v_1, \dots, v_n])[e_{i_0}^n, \dots, e_{i_{n-1}}^n] \\ &= ff[v_1[e_{i_0}^n, \dots, e_{i_{n-1}}^n] = e_{j_0}^n, \dots, v_n[e_{i_0}^n, \dots, e_{i_{n-1}}^n]] \\ &= ff[e_{j_0}^n, \dots, e_{j_{n-1}}^n] \\ &= s \\ &= \sigma_s(f). \end{aligned}$$

Therefore, $\sigma_v o_{coh} \sigma_t = \sigma_s$.

Similarly, one can show that $\sigma_t = \sigma_u o_{coh} \sigma_s$ for some $\sigma_u \in Cohyp^{lin}(n)$.

Hence, $\sigma_t L \sigma_s$.

Remark The number of pairs (σ_t, σ_s) in which $\sigma_t L \sigma_s$ is $n + (n!)^2$.

Next, we characterize the Green's relation H on $Cohyp^{lin}(n)$.

Theorem 2.7 Let $\sigma_t, \sigma_s \in Cohyp^{lin}(n)$ and $t, s \in \{e_i^n \mid n \geq 1, 0 \leq i \leq n-1\}$. Then $\sigma_t H \sigma_s$ if and only if $t = s$.

Proof Assume that $\sigma_t H \sigma_s$.

Then $\sigma_t H \sigma_s$ and $\sigma_t R \sigma_s$.

By Theorem 2.4, we get that $t = s$.

Similarly, assume that $t = s$.

Then $\sigma_t = \sigma_s$.

Since and are equivalence relations,

we have $\sigma_t L \sigma_s$ and $\sigma_t R \sigma_s$.

Therefore, $\sigma_t H \sigma_s$.

Theorem 2.8 Let $t, s \in cT_t^{lin,(n)} \setminus \{e_i^n \mid n \geq 1, 0 \leq i \leq n-1\}$. Then $\sigma_t H \sigma_s$.

Proof Let $t = ff[e_{i_0}^n, \dots, e_{i_{n-1}}^n] \in cT_t^{lin,(n)}$ and $s = ff[e_{j_0}^n, \dots, e_{j_{n-1}}^n] \in cT_s^{lin,(n)}$ for $i_0, \dots, i_{n-1}, j_0, \dots, j_{n-1} \in \{0, \dots, n-1\}$.

By Theorem 2.2, we have that $\sigma_t R \sigma_s$.

By Theorem 2.6, we have that $\sigma_t L \sigma_s$.

Therefore, $\sigma_t H \sigma_s$.

Remark The number of pairs (σ_t, σ_s) in which $\sigma_t H \sigma_s$ is $n + (n!)^2$.

Next, we characterize the Green's relation D on $Cohyp^{lin}(n)$.

Theorem 2.9 Let $(\sigma_t, \sigma_s) \in Cohyp^{lin}(n)$ and $t, s \in \{e_i^n \mid n \geq 1, 0 \leq i \leq n-1\}$. Then $\sigma_t D \sigma_s$.

Proof Since $\sigma_t L \sigma_t$ and by Theorem 2.2,

we have that $\sigma_t R \sigma_t$.

Then $\sigma_t D \sigma_t$.

Theorem 2.10 Let $t, s \in cT_t^{lin,(n)} \setminus \{e_i^n \mid n \geq 1, 0 \leq i \leq n-1\}$. Then $\sigma_t D \sigma_s$.

Proof Let $t = ff[e_{i_0}^n, \dots, e_{i_{n-1}}^n] \in cT_t^{lin,(n)}$ and $s = ff[e_{j_0}^n, \dots, e_{j_{n-1}}^n] \in cT_s^{lin,(n)}$ for $i_0, \dots, i_{n-1}, j_0, \dots, j_{n-1} \in \{0, \dots, n-1\}$.

By Theorem 2.2, we have that $\sigma_t R \sigma_s$.

By Theorem 2.6, we get that $\sigma_t L \sigma_s$.

Therefore, $\sigma_t D \sigma_s$.

For linear cohypersubstitutions σ_t, σ_s such that t and s are different form i.e., $t \in \{e_i^n \mid 0 \leq i \leq n-1\}$ and $s \in cT_t^{lin,(n)} \setminus \{e_i^n \mid 0 \leq i \leq n-1\}$, we have that $(\sigma_t, \sigma_s) \notin D$ as the following example:

Example 2.11 Let $\sigma_t, \sigma_s \in Cohyp^{lin}(n)$ and $t = e_i^n$, $s = ff[e_{j_0}^n, \dots, e_{j_{n-1}}^n] \in cT_t^{lin,(n)}$ for all $i, j_0, \dots, j_{n-1} \in \{0, \dots, n-1\}$ and $E(s)$ be distinct.

Then $\sigma_t(f) = e_i^n$ and $\sigma_s(f) = ff[e_{j_0}^n, \dots, e_{j_{n-1}}^n]$.

By Theorem 2.4, we get that $\sigma_t L \sigma_s$.

But by Theorem 2.3, we have that $(\sigma_t, \sigma_s) \notin R$.

Hence, $(\sigma_t, \sigma_s) \notin D$.

Remark The number of pairs (σ_t, σ_s) in which $\sigma_t D \sigma_s$ is $n^2 + (n!)^2$.

Next, we characterize the Green's relation J on $Cohyp^{lin}(n)$.

Theorem 2.12 Let $(\sigma_t, \sigma_s) \in Cohyp^{lin}(n)$ and $t, s \in \{e_i^n \mid n \geq 1, 0 \leq i \leq n-1\}$. Then $\sigma_t J \sigma_s$.

Proof Let $t = e_i^n, s = e_j^n$ and $u \in cT_t^{lin,(n)}$.

Since $\hat{\sigma}_u[e_k^n] = e_k^n$ for all $k = 0, \dots, n-1$.

we have

$$\sigma_t(f) = e_i^n$$

$$= \hat{\sigma}_s[e_i^n]$$

$$= \hat{\sigma}_u[\hat{\sigma}_s[e_i^n]]$$

$$= \hat{\sigma}_u[\hat{\sigma}_s[\sigma_{e_i^n}(f)]]$$

$$= \hat{\sigma}_u[\hat{\sigma}_s[\sigma_t(f)]]$$

Therefore, $\sigma_t = \sigma_u \circ_{coh} \sigma_s \circ_{coh} \sigma_t$.

Similarly, one can show that $\sigma_s = \sigma_x \circ_{coh} \sigma_t \circ_{coh} \sigma_y$ for some $\sigma_x, \sigma_y \in Cohyp^{lin}(n)$.

Hence, $\sigma_t J \sigma_s$.

Theorem 2.13 Let $t, s \in cT_t^{lin,(n)} \setminus \{e_i^n \mid n \geq 1, 0 \leq i \leq n-1\}$. Then $\sigma_t J \sigma_s$.

Proof Let $t = ff[e_{i_0}^n, \dots, e_{i_{n-1}}^n] \in cT_t^{lin,(n)}$ and $s = ff[e_{j_0}^n, \dots, e_{j_{n-1}}^n] \in cT_s^{lin,(n)}$ for $i_0, \dots, i_{n-1}, j_0, \dots, j_{n-1} \in \{0, \dots, n-1\}$.

We let $r = ff[r_1, \dots, r_n]$ such that $r_{j_k} = e_{j_k}^n$ where $j_k \in \{0, \dots, n-1\}$ and $k = 0, \dots, n-1$.

By Theorem 2.2, we get that $\sigma_t(f) = \hat{\sigma}_s[\sigma_r(f)]$.

Let $v = (f)[e_{i_0}^n, \dots, e_{i_{n-1}}^n] \in cT_t^{lin,(n)}$.

Then

$$\hat{\sigma}_v[\sigma_t(f)] = \hat{\sigma}_v[ff[e_{i_0}^n, \dots, e_{i_{n-1}}^n]]$$

$$= \sigma_v(f)[e_{i_0}^n, \dots, e_{i_{n-1}}^n]$$

$$= (ff[v_1, \dots, v_n])[e_{i_0}^n, \dots, e_{i_{n-1}}^n]$$

$$= ff[v_1[e_{i_0}^n, \dots, e_{i_{n-1}}^n] = e_{j_0}^n, \dots, v_n[e_{i_0}^n, \dots, e_{i_{n-1}}^n]]$$

$$= ff[e_{j_0}^n, \dots, e_{j_{n-1}}^n]$$

$$= t$$

$$= \sigma_t(f)$$

Therefore, $\sigma_v \circ_{coh} \sigma_s \circ_{coh} \sigma_r = \sigma_t$.

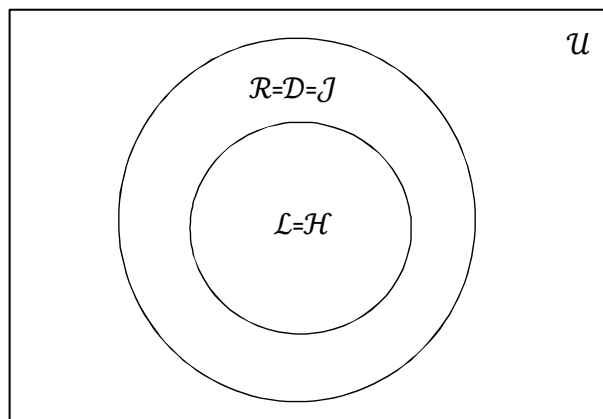
Similarly, one can show that $\sigma_s = \sigma_x \circ_{coh} \sigma_t \circ_{coh} \sigma_y$ for some $\sigma_x, \sigma_y \in Cohyp^{lin}(n)$.

Hence, $\sigma_t J \sigma_s$.

Remark The number of pairs (σ_t, σ_s) in which $\sigma_t J \sigma_s$ is $n^2 + (n!)^2$.

We conclude the R, L, H, D and J as the following diagram:

$$U = \{(\sigma_t, \sigma_s) \mid t, s \in cT_t^{lin,(n)}\}$$



If $t, s \in \{e_i^n \mid n \geq 1, 0 \leq i \leq n-1\}$ and $t = s$ in L , then $L \subseteq R$.

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References

1. Drbohlav K. On quasicvarieties. Acta Fac. Rerum Natur. Univ. Comenian. Math. Mimoriadne Cislo. 1971 ; 17-20.
2. Csa'ka'ny B. Completeness in coalgebras. Acta Sci. Math. 1985 ; 48: 75-84.
3. Denecke K, Saengsura K. Menger Algebras and Clones of Cooperations. Algebra Colloquium. 2008 ; 15(2): 223-234.
4. Boonchari D, Saengsura K. Monoid of Cohypersubstitutions of type $t = (n)$. Thai Journal of Mathematics. 2016 ; 14: 191-201.
5. Denecke K. The partial clone of linear terms. Siberian Mathematical Journal. 2016 ; 57(4): 589-598
6. Denecke K, Lau D, Poschel R, Schweigert D. Hyperidentities, hyperequational classes and clone congruences. Contribution to General Algebra. 2002 ; 7: 97-118.
7. Denecke K, Saengsura K. Cohyperidentities and M-solid classes of coalgebras. Discrete Mathematics. 2009 ; 304(4): 772-783.

8. Denecke K, Wismath SL. Universal algebra and applications in theoretical computer science. Boca Raton, Chapman&Hall/CRC. 2002.
9. Howie JM. Fundamentals of Semigroup Theory. Oxford Science Publications, Clarendon Press. Oxford. 1995.
10. Jermjitporn S, Saengsura N. Generalized cohypersubstitutions of type $\cdot = (\cdot)_{i \in I}$. Thai Journal of Mathematics. 2013 ; 4: 747-755.
11. Koppitz J, Denecke K. M-solid varieties of algebras. Springer Science+Business Media. Inc. 2006.
12. Leeratanavalee S, Denecke K. Generalized hypersubstitutions and strongly solid varieties. Proceedings of the "59th Workshop on General Algebra", Conference for Young Algebraists. Potsdam. 200 ; 135-145.