การกระเจิงของศักยยูกาวาโดยการใชหลักการควอนตัมเชิงพลวัต Yukawa scattering treated by the Quantum dynamical principle

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บทคัดยอ

การกระเจิงโดยศักยยูกาวา ถูกยกเปนกรณีการศึกษา ผานการใชเทคนิคการคํานวณโดยหลักการเชิงพลวัตควอนตัมที่เสนอ โดยชวิงเงอร์ ซึ่งเป็นวิธีการที่ขึ้นกับฟังก์ชันกำเนิดที่ถูกแทนที่ด้วยตัวกระทำการแบบฟังก์ชันเชิงอนุพันธ์ จากผลลัพธ์เราได้ ลักษณะอซิมโทติกของกรีนฟงกชันอิสระ ที่สามารถอธิบายลักษณะของการกระเจิงของอนุภาคตอศักยยูกาวา และทําการแปรคา พารามิเตอร์ของมวลให้มีค่าต่างๆ กัน นอกจากนี้ผลลัพธ์ที่ได้นี้ยังนำไปสู่ค่าแอมพลิจูดของการกระเจิงและค่าภาคตัดขวางของ การกระเจิงเชิงอนุพันธอันเนื่องมาจากศักยยูกาวาอีกดวย

คําสําคัญ: หลักการควอนตัมเชิงพลวัต การกระเจิงโดยศักยยูกาวา ศักยระยะสั้น กรีนฟงกชัน

Abstract

Yukawa scattering is pedagogically interpreted, by the Schwinger's quantum dynamical principle involving the generating function, which is replaced by a functional differential operation. As for the results, we get the asymptotically free Green function that explains the behavior of the Yukawa potential when the mass parameter is increasing and it can also lead to scattering amplitude and differential cross section respectively.

Keywords: quantum dynamical principle, Yukawa scattering, short range potentials, Green functions.

Introduction

In quantum scattering, we are interested in an interaction between the incident particles and the potential of the target e.g., coulomb potential¹ $V(x) = 1/x$ which describes the behavior of particle scattering. Yukawa² presented his study by considering the meson interaction, particle with mass, which eventually was called the Yukawa potential,. Experimentally, researchers studied the scattering amplitude to determine these scattered particles. R. Feynman presented a diagram of particle

scattering with the path integral that uses the time-slicing derivation $3,4$.

 Accordingly, in this report, we use the quantum dynamical principle proposed by J. Schwinger⁵⁻⁹ to describe this situation. This method is very useful because it gives us the interested transformation function, also called the propagator. In particular, the Hamiltonian equation of this system involves external sources which generate degrees of freedom $10,11,12$. The equation is precisely derived from the variation of the transformation

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function that depends on the potential, and then it is replaced by the differential functional. Consequently, it leads to the scattering amplitude. the scattering amplitude. the scattering amplitude. T_{tot} and T_{tot} amplitude. leads to the scattering amplitude. \mathbf{F}

The main purpose of this paper is to find the The main purpose of this paper is to find scattering amplitude and differential cross section by evaluating the asymptotically free Green function through the Yukawa potential. Previously, this method was also used to explain Coulomb scattering^{13,14} near an energy shell. Clearly, this paper also shows the process, by Clearly, this paper also shows the process, by Clearly, this paper also shows the process, by setting tools, for interpreting the scattering problem in quantum theory by using the Yukawa potential which is involved with the mass term. scattering amplitude and differential cross section I quantum theory by using the Yukawa potential which is involved with the mass term. The main purpose of this paper is to find the show, cloudy, the puper died choice the process, $\frac{1}{2}$ The main purpose of this paper is to find the $t_{\rm{max}}$ and differential cross \sim scalushing amplitude and unterential cros shell. Clearly, this paper also shows the process, by setting tools, for interpreting the scattering problem p_{v} are p_{v} in p_{v} in p_{v} with p_{v} with p_{v} with p_{v} and p_{v} patham moory by doing the randward \mathcal{I} integrating Eq. (4) over \mathcal{I} the main parpose of and paper scattering amplitude and differential cr problem in the problem in quantum theory by using the Yukawa was the Wukawa was the Wuk potential which is involved with the mass term. **Quantum dynamical principle for scattering case Quantum dynamical principle for scattering case** scattering amplitude and differential cross section by Predicting unphead and also brief of explaining the state of the explanation of the expla evaluating the asymptotically free Green function through Previously, this method was also used to explain evaluating the asymptotically free Green function through the main purpose of this paper is to find $\frac{1}{\sqrt{2}}$. This method was also used to this method was also used to explain $\frac{1}{\sqrt{2}}$ Coulomb scattering13,14 near an energy shell. setting tools, for interpreting the scattering setting tools, for interpreting the scattering earlier. potting tools, for interpretting the coultering provide $\sqrt{2}$ $\frac{d}{dx}$. The main purpose of this paper is to find the scattering amplitude and differential cross section by evaluating the asymptotically free Green function through Previously, this method was also used to explain scattering amplitude and differential cross sed

Quantum dynamical principle for scattering case nical principle for scattering case *y*namical principle for scatteri<mark>r</mark>

We start with a typical Hamiltonian written as with a typical Hamiltonian written as
v v e start with a typical Hamiltonian written as where α (ypical α)

$$
H = \frac{\mathbf{p}^2}{2m} + V(\mathbf{x}),\tag{1}
$$

where \bm{p} is the momentum of a particle with mass m and incident on a potential $\mathit{V}(\mathbf{x})$. Furthermore, we present a new Hamiltonian $\text{H} \left(\lambda, \tau \right)$ as follows by $\frac{1}{\sqrt{1-\frac{1}{2}}}$ and incident on a potential $\mathit{V}(\mathbf{x})$. Furthermore, we pi

$$
H(\lambda, \tau) = \frac{\mathbf{p}^2}{2m} + \lambda V(\mathbf{x}) - \mathbf{x} \cdot \mathbf{F}(\tau) + \mathbf{p} \cdot \mathbf{S}(\tau),
$$
 (2)

The latter involves the external sources $\, {\bf F}(\tau) \,$ and $\, {\bf S}(\tau) \,$ at time τ . These sources are linear function of **x** and $\mathbf p$. The sources generate $\mathbf x(\tau)$ and $\mathbf p(\tau).$ for position and momentum at time τ respectively. The parameter $\langle x | p t' \rangle^0 = \exp \left[-\frac{i}{2m\hbar}\int_{t'}^{t'}$ and momentum at time τ respectively. The parameter λ is an arbitrary parameter. The arbitrary parameter is another physical quantity involved with the system that we don't need to specify. It will be eventually set equal eventually set experimentally set equal to one (because of the boundary condition of the transformation function). The latter involves the external sources $\, {\bf F}(\tau) \,$ and $\, {\bf S}(\tau) \,$ derivation3,4. to one (because of the boundary condition of the menton inductive it gives useful because it gives useful because it gives useful because the contract of the c condition of the

Next we introduce Schwinger's quantum dynamical principle in the variation of transformation function from \bf{p} at time t' (initial state) to \bf{x} at time t (final state), written as Next we introduce Schwinger's quantum dynamical principle in the variation of transformation Next we introduce Schwinger's quantum dynamical principle in the variation of transformation Next we introduce Schwinger's quantum function from **p** at time $\frac{1}{2}$ (in time $\frac{1}{2}$ state) to $\frac{1}{2}$ (in time $\frac{1}{2}$ state) to Next we introduce Schwinger's quantum dynamical principle in the variation of transformation Next we introduce Schwinger's quantum dynamical principle in the variation of transformation next were introduced in the constant of the extension of dynamical principle in the variation of transformation Next we introduce Schwinger's quantum dynamical principle in the variation of transformation Next we introduce Schwinger's quantum dynamical principle in the variation of transformation

$$
\delta \langle \mathbf{x} t | \mathbf{p} t' \rangle = -\frac{i}{\hbar} \int_{t'}^{t} d\tau \langle \mathbf{x} t | \delta \mathbf{H} (\mathbf{x}(\tau), \mathbf{p}(\tau), \tau; \lambda) | \mathbf{p} t' \rangle. \qquad (3)
$$

$$
\langle \mathbf{x} t | \mathbf{p} t' \rangle_{0} = \mathbf{e} \mathbf{x}
$$

5 This leads to $|$ ds to $|$, $|$, $|$, $|$, $|$, $|$, $|$, $|$, $|$, $|$, $|$, $|$, $|$, $|$, $|$, $|$, $|$, $|$, $|$, $|$, $|$, $|$, $|$, $|$, $|$, $|$, $|$, $|$, $|$, $|$, $|$, $|$, $|$, $|$, $|$, *t to <i>t***_{***i***}** *<i>t***_{***i***** *<i>t***_{***i***}** *<i>t***_{***i***** *<i>t***_{***i***}** *<i>t***** *<i>t*****}}</sub></sub></sub> *x* **p x** *x* **p F***x x* *****x x* This leads to

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uently, it
find the

$$
\delta \langle \mathbf{x}t | \mathbf{p}t' \rangle = -\frac{i}{\hbar} \int_{t'}^{t} d\tau \delta \left[\lambda V \left(-i\hbar \frac{\delta}{\delta \mathbf{F}(\tau)} \right) \right] \langle \mathbf{x}t | \mathbf{p}t' \rangle, \quad (4)
$$

by inserting the Hamiltonian from Eq. (2) into Eq. (3). So,
tion through ab also this variation satisfies this Hamiltonian and depends on
od was also r an energy the parameter λ . In addition, for $V(\mathbf{x})$, \mathbf{x} is replaced Fan energy
process, by by $-i\hbar\delta/\delta{\bf F}(\tau)$, which was denoted earlier. by
by inserting the Hamiltonian from Eq. (2) into Eq. (3). So. $\log h$ this variation satisfies this Hamiltonian and depends on **2** is represented a light differential $V(\mathbf{x})$ **x** is replaced by $\boldsymbol{\mu}$ is replaced $\boldsymbol{\mu}$, in addition, for \boldsymbol{r} (**A**), **A** is the narlier. ру $-\bar{l}\hbar\partial$ / $\partial\mathbf{F}(\tau)$, which was denoted earlier.

> $\mathcal{I} = \mathcal{I}$ Immediately, integrating $\lambda = 0$ to 1, we obtain in Immediately, integrating Eq. (4) over λ from $\lambda = 0$ to 1, we obtain $\overline{}$ $\overline{\$

Case
\n
$$
\langle \mathbf{x} | \mathbf{p} \mathbf{r}' \rangle = \exp \left[-\frac{i}{\hbar} \int_{\mathbf{r}}^{\mathbf{r}} d\tau V \left(-i\hbar \frac{\delta}{\delta \mathbf{F}(\tau)} \right) \right] \langle \mathbf{x} | \mathbf{p} \mathbf{r}' \rangle^0 |_{\mathbf{F} = 0, \mathbf{S} = 0},
$$
\n(5)
\n**Case**
\n
$$
\mathbf{a} \mathbf{s} \qquad \text{where } \langle \mathbf{x} \mathbf{f} | \mathbf{p} \mathbf{f}' \rangle \text{ satisfied the Hamiltonian in Eq. (1) and}
$$

setting the parameter $\lambda = 1$. The transformation function (1) $\langle \mathbf{x}t | \mathbf{p}t' \rangle^0$ is governed by the free Hamiltonian. The source terms, $F(\tau)$ and $S(\tau)$ are finally set equal to cle with mass $\,m_{\rm{z}}\,$ zero to satisfy the Hamiltonian in Eq. (1). Zero-superscript thermore, we pre-

denotes the free particle where it's Hamiltonian is given by it's Hamiltonian is given by \mathcal{L} re
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t expelte *^t ^V ⁱ* Itonian. The
set equal to by by **e**
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erms, $\mathbf{F}(\tau)$ and $\mathbf{S}(\tau)$ are finally set e
atisfy the Hamiltonian in Eq. (1). Zero-sup *s* governed by the free Hamiltonian. The
 t, $\mathbf{F}(\tau)$ and $\mathbf{S}(\tau)$ are finally set equal to
 t the Hamiltonian in Eq. (1). Zero-superscript

free particle where it's Hamiltonian is given
 $\frac{\mathbf{p}^2}{2m} - \mathbf{x} \$ source terms, $\mathbf{F}(\tau)$ and $\mathbf{S}(\tau)$ are finally set equal to y $\mathsf{b}\mathsf{v}$ \mathbf{u} **F** $\frac{1}{2}$ **S** $\frac{1}{2}$ dy

$$
(2) \qquad \mathcal{H}(0,\tau) = \frac{\mathbf{p}^2}{2m} - \mathbf{x} \cdot \mathbf{F}(\tau) + \mathbf{p} \cdot \mathbf{S}(\tau). \tag{6}
$$

and $S(\tau)$ By applying the quantum dynamical principle to Eq.(6) $\mathbf{F}(\tau)$ and $\mathbf{S}(\tau)$ and epiacing \mathbf{p} by $i\hbar \delta / \delta \mathbf{S}(\tau)$, we obtain
 $\mathbf{p}(\tau)$, for position *t x* \mathbf{p} \mathbf{q} \mathbf{r} (6) **x p** *t t* | is a *transformation function* of *t t* ' . **x** and replacing **p** by $i\hbar \delta / \delta S(\tau)$, we obtain $\mathbf{f}(\mathbf{z})$ **x p** *t t* | is a *transformation function* of *t t* ' .

$$
\mathbf{p}(\tau)
$$
, for position
by. The parameter $\langle \mathbf{x}t | \mathbf{p}t' \rangle^0 = \exp \left[-\frac{i}{2m\hbar} \int_t^t d\tau \left(i\hbar \frac{\delta}{\delta S(\tau)} \right)^2 \right] \langle \mathbf{x}t | \mathbf{p}t' \rangle_0.$ (7)

h the system that The transformation function $\langle \mathbf{x}t | \mathbf{p}t' \rangle_0$ satisfies the entually set equal Hamiltonian equation, i.e. without the kinetic term, which of the is defined as which es uie es the differential constants the differential constants of the differen section by using the *asymptotically free Green* section by using the *asymptotically free Green* $rac{1}{2}$ $\frac{1}{\pi}$ is defined as section by using the *asymptotically free Green*

$$
H_0(\tau) = -\mathbf{x} \cdot \mathbf{F}(\tau) + \mathbf{p} \cdot \mathbf{S}(\tau).
$$
 (8)

Transie ι by the way, the *Heisenberg equation* of the at time t $\;\;\;\;$ By the way, the Heisenberg equation of the special s_j is at all and $s = 2j$ are any, are measured grounded. Finally, we obtain By the way, the *Heisenberg equation* of the $\frac{1}{2}$ is $\frac{1}{2}$ is $\frac{1}{2}$ integrated. Finally, which is integrated. Finally, we have $\frac{1}{2}$ *equation* $\ddot{\ }$ By the way, the *Heisenberg equation* of the By the way, the *Heisenberg equation* of the special nam Hamiltonian in Eq. (8) is integrated. Finally, we obtain by the way, the Heisenberg equation of the special

(3)

$$
\langle \mathbf{x}t | \mathbf{p}t' \rangle_0 = \exp\left[\frac{i}{\hbar}\mathbf{x} \cdot \left(\mathbf{p} + \int_{t'}^{t} d\tau \mathbf{F}(\tau)\right)\right]
$$

$$
\times \exp\left[-\frac{i}{\hbar}\mathbf{p} \cdot \int_{t'}^{t} d\tau \mathbf{S}(\tau)\right]
$$

$$
\times \exp\bigg[-\frac{i}{\hbar}\int_{t'}^{t} d\tau \int_{t'}^{t} d\tau' S(\tau)\Theta(\tau-\tau')F(\tau')\bigg],\tag{9}
$$

where the Heaviside step function, $\Theta(\tau)$, is used as the time controller. In particular, from Eq. (9), when we set $\mathbf{S}(\tau)$ and $\mathbf{F}(\tau)$ equal to zero and substitute into Eq. (7) thus Eq. (5) is directly rewritten as $\sum_{i=1}^{n}$ is directly rewritten as $\sum_{i=1}^{n}$ and 0.000 minutes of 0.00
and 0.000 minutes of 0.00
 \mathcal{L} q. (*b*) to directly formation do $\begin{aligned} \mathsf{L} \mathsf{S} \mathsf{S} \mathsf{S} \mathsf{S} \end{aligned}$ is directly rewritten as \mathbf{w} **x** F_{**x**} **x** \mathbf{w} **x** $\$ **^p x F ,** where the Heavisid \Rightarrow d where the Heaviside step function, Θ

$$
\langle \mathbf{x}t | \mathbf{p}t' \rangle = \exp\left[\frac{i}{\hbar} \left(\mathbf{x} \cdot \mathbf{p} - \frac{\mathbf{p}^2}{2m} (t - t') \right) \right]
$$

$$
\times \exp\left[\frac{i\hbar}{2m} \int_{t'}^{t} d\tau \int_{t'}^{t} d\tau' [t - \tau_{>}] \frac{\delta}{\delta \mathbf{F}(\tau)} \cdot \frac{\delta}{\delta \mathbf{F}(\tau')} \right]
$$

$$
\times \exp\left[-\frac{i}{\hbar} \int_{t'}^{t} d\tau V \left(\mathbf{x} - \frac{\mathbf{p}}{m} (t - \tau) + \mathbf{F}(\tau) \right) \right] |_{\mathbf{F} = 0, \mathbf{S} = 0}, \quad (10)
$$

potential function, $-i\hbar \delta / \delta \mathbf{F}(\tau)$ by $\mathbf{x} - \mathbf{p}(t-t') / m$ to get Eq. (10). Finally, we get the translational invariant $u = \frac{\alpha}{\hbar} [p^0 - E(\mathbf{p}^0)].$ \mathbf{r} by Eq. (10). Finally, we get the translational life in time when setting $\mathbf{F}(\tau) = 0$. $\langle \mathbf{x}t | \mathbf{p}t' \rangle$ is a transfor-
Thus Eq. (13) becomes mation function of $t-t'$. where τ is maximum of τ and τ' . We replace, for the integration variable as translational invariant in time when setting **F**() 0 . *x* where τ_{S} is maximum of *t* and *t* . We replace **x p** *t t* | is a *transformation function* of *t t* ' . **x p** *t t* | is a *transformation function* of *t t* ' . $M_{\rm H}$ and $M_{\rm H}$ are differential cross the differential cross the differential cross the differential cross that $M_{\rm H}$ **The differential cross section from the** to get Eq. (TO). Finally, we get the trans section by using the *asymptotically free Green* Potomar famolion, $\mathbf{v}_1 \mathbf{v}$ or (\mathbf{v}) by \mathbf{v} to get Eq. (TO). Finally, we get the trans section by using the *asymptotically free Green*

totically free Green function The differential cro The differential cross section from the asymp- $G_+(\mathbf{p},\mathbf{p}^{\prime};p^0)[p^0-\mathbf{p}^{\prime}]$ **asymptotically free Green function** next, we differential cross constitution from the consum *f* the differential cross section from the asym *function* which is a function at infinite time. *function* which is a function at infinite time. *function* which is a function at infinite time.

Next, we determine the differential cross section by using the *asymptotically free Green function* which is $\begin{pmatrix} 1 & y \\ z & z \end{pmatrix}$ a function at infinite time. **asymptotically free Green function** a function at infinite time. $\frac{1}{2}$ and $\frac{1}{2}$ i.e. $\frac{1}{2}$ i.e. $\frac{1}{2}$

We recall the definition of Green function that *functional the domination* of Croom rank section by using the *asymptotically free Green formation* which is a function at the function $\frac{1}{2}$ section by using the *asymptotically free Green for functional inc. function* which is a function at θ function at θ We recall the definition of Green function that

$$
\langle \mathbf{x}t | \mathbf{p}t' \rangle = G_{+}(\mathbf{x}t, \mathbf{p}t') = \int d^{3}\vec{x}' e^{i\mathbf{x}' \cdot \mathbf{p}/\hbar} G_{+}(\mathbf{x}t, \mathbf{x}'t') ,
$$
(11)

for $G_{+}(\mathbf{x}t, \mathbf{x}^{T}t')$ is denoted as $\langle \mathbf{x}t | \mathbf{x}^{T}t' \rangle$. *t t G tt* | (,) **x p xp** σ_{+} ¹ $W(x, y)$ is defined to $\sum_{i=1}^{n} V_i$. \mathbf{A} , \mathbf{A} \mathbf{v} \mathbf{y} is defined as \mathbf{A} \mathbf{v} \mathbf{A} \mathbf{v} \mathbf{y} .

the Fourier tran θ we for the disc tile Fourier transform to fee We use the Fourier transform to rewrite Eq. (11)

 3 '/ d (,') *ⁱ xe G t t* **x x** $\frac{1}{2}$ d ³ $\frac{1}{2}$ $\frac{1$ **x** $\frac{1}{2}$ **x** $\frac{1}{2}$ as

$$
G_{+}(\mathbf{p}, \mathbf{p}'; p^0) = -\frac{i}{\hbar} \frac{1}{(2\pi\hbar)^3} \int_0^\infty d\alpha e^{i(p^0 + i\omega)\alpha/\hbar}
$$
\nClearly, Eq. (16) can be rew
\n
$$
\times \int d^3 \mathbf{x} e^{-i\mathbf{x} \cdot \mathbf{p}/\hbar} \langle \mathbf{x} \alpha | \mathbf{p}' \mathbf{0} \rangle,
$$
\n(12)
$$
G_{+}(\mathbf{p}, \mathbf{p}'; p^0) [p^0 - E(\mathbf{p}')
$$

 $\frac{1}{2}$ $\langle X\alpha | \mathbf{p} \cdot \mathbf{0} \rangle$ is given in Eq. (10) with $\iota =$
 $\rightarrow +0$. $\mathbf{M} = \langle \mathbf{X} \mathbf{Z} | \mathbf{p} \rangle$ is given in and $\omega \rightarrow +0$. $\begin{align} \mathbf{x} \times \mathbf{y} &\infty \setminus \mathbf{p} \setminus \mathbf{0}, \ \mathbf{x} \mid \mathbf{p} \setminus \mathbf{0} &\infty \end{align}$ is given in Eq. (10) with $t-t' \equiv \alpha$ where $\langle x\alpha | p'0 \rangle$ is given in Eq. (10) with $t-t' \equiv \alpha$,

Next, insert Eq. (10) into Eq. (12), we obtain

(9)
$$
G_{+}(\mathbf{p}, \mathbf{p}'; p^0) = -\frac{i}{\hbar} \frac{1}{(2\pi\hbar)^3} \int_0^{\infty} d\alpha e^{i(p^0 - E(\mathbf{p}') + i\omega)\alpha/\hbar}
$$

is used as the
when we set $\times \int d^3 \vec{x} e^{-i\mathbf{x} \cdot (\mathbf{p} - \mathbf{p}')/\hbar} K(x, \mathbf{p}'; \alpha)$, (13)
te into Eq. (7)

where $E(\mathbf{p})$ The function $K(\lambda, \mu, \alpha)$ is defined as $F(r) = \frac{12}{2}$ $F(r) = \frac{12}{2}$ where $E(\mathbf{p}') = \mathbf{p}'^2 / 2m$. The function $K(\mathbf{x}, \mathbf{p}'; \alpha)$ is denoted as $E(n^2 - n^2^2/2m)$ where $E(\mathbf{P}') = \mathbf{P}' \cdot Zm$.
The function $K(\mathbf{x} | \mathbf{n}' \cdot \alpha)$ is denoted as where $E(\mathbf{p}) - \mathbf{p}$ / $2m$.

 $N_{\rm eff}$ insert Eq. (10) into Eq. (10) into Eq. (12) , we obtain

$$
K(\mathbf{x}, \mathbf{p}'; \alpha) = \exp\left[\frac{i\hbar}{2m} \int_{t'}^{t} d\tau \int_{t'}^{t} d\tau' [t - \tau_{>}] \frac{\delta}{\delta \mathbf{F}(\tau)} \frac{\delta}{\delta \mathbf{F}(\tau')}\right]
$$

$$
\times \exp\left[-\frac{i}{\hbar} \int_{t'}^{t} d\tau V \left(\mathbf{x} - \frac{\mathbf{p}'}{m}(t - \tau) + \mathbf{F}(\tau)\right)\right]\Big|_{\mathbf{F} = 0}.
$$
 (14)

energy shell $p^0 \approx p'^2 / 2m$. So, we introduce the new integration variable as **produced** $F^2 + \langle \mathbf{F}^3 \mathbf{F}^3 \rangle^2$
shell $n^0 \simeq n^{12}/2m$. So we introduce the new for the integration variable as For $\alpha \rightarrow \infty$, we consider the $G_{+}(\mathbf{p}, \mathbf{p}'; p^0)$ near the energy shell $p^0 \approx p^{12}/2m$. So, we introduce the new
for the integration variable as the new integration variable as where \mathbb{R} cos ' / | '|| | **px p x** . Eq. (19) is independent of *u* cos ' / | '|| | **px p x** . Eq. (19) is independent of *u* where is Euler-Mascheroni constant and the new integration variable as

$$
u = \frac{\alpha}{\hbar} [p^0 - E(\mathbf{p}^0)].
$$
\nThus Eq. (13) becomes

 T_{max} becomes T_{max} Thus Eq. (13) becomes $\frac{1}{2}$ becomes Eq. (13) becomes

on from the asymp-
\n
$$
G_{+}(\mathbf{p}, \mathbf{p}'; p^0)[p^0 - E(\mathbf{p}')] = -\frac{i}{(2\pi\hbar)^3} \int_0^\infty due^{iu(1+i\omega)}
$$
\nDifferential cross section
\nGreen function which is\n
$$
\times \int d^3x e^{-ix \cdot (\mathbf{p} - \mathbf{p})/\hbar} K\left(\mathbf{x}, \mathbf{p}'; \frac{u\hbar}{p^0 - E(\mathbf{p}')} \right).
$$
\n(16)

set $F(\tau) = 0$, and obtain Considering only $K(x, p, \omega h)$ $p = E(p)$ where $\mathbf{F}(t) = 0$, and obtain that Considering only $K(\mathbf{x}, \mathbf{p}'; u \hbar / p^0 - E(\mathbf{p}'))$ where $\mathbf{r} \times \mathbf{r} \times \mathbf{$ that Considering only $K(\mathbf{x}, \mathbf{p}'; u \hbar / p^0 - E(\mathbf{p}'))$ where we
set $\mathbf{F}(\tau) = 0$, and obtain

set
$$
F(\tau) = 0
$$
, and obtain
\n
$$
K\left(\mathbf{x}, \mathbf{p}'; \frac{u\hbar}{p^0 - E(\mathbf{p}')} \right)
$$
\n
$$
\approx \exp\left[-\frac{i}{\hbar} \int_0^{u\hbar/(p^0 - E(\mathbf{p}^0))} d\alpha V \left(\mathbf{x} - \frac{\mathbf{p}'}{m} \alpha \right)\right]
$$
(17)

 $C = C \cup C$, $C = C$,

the into Eq. (7)
\nwhere
$$
E(\mathbf{p}') = \mathbf{p}^{12}/2m
$$
.
\nThe function $K(\mathbf{x}, \mathbf{p}'; \alpha)$ is denoted as
\n
$$
K(\mathbf{x}, \mathbf{p}'; \alpha) = \exp\left[\frac{i\hbar}{2m}\int_{r}^{t} d\tau \int_{r}^{t} d\tau [\tau - \tau_{2}] \frac{\delta}{\delta F(\tau)} \frac{\delta}{\delta F(\tau)} \right]
$$
\n
$$
\times \exp\left[-\frac{i}{\hbar}\int_{r}^{t} d\tau V \left(\mathbf{x} - \frac{\mathbf{p}'}{m}(t-\tau) + F(\tau)\right) \right]|_{F=0}. \quad (14)
$$
\n
$$
|\mathbf{r}_{=0,5=0}, (10) \quad \text{For } \alpha \to \infty, \text{ we consider the } G_{+}(\mathbf{p}, \mathbf{p}'; p^{0}) \text{ near the energy shell } p^{0} = p^{12}/2m. \text{ So, we introduce the new\nreplace, for the integration variable as\n $\mathbf{p}(t-t')/m$
\n $u = \frac{\alpha}{\hbar} [p^{0} - E(\mathbf{p}')] . \qquad (15)$
\nis a *transfor*.
\nThus Eq. (13) becomes
\nthe **asymp-** $G_{+}(\mathbf{p}, \mathbf{p}'; p^{0}) [p^{0} - E(\mathbf{p}')] = -\frac{i}{(2\pi\hbar)^{3}} \int_{0}^{\infty} d\mu e^{i\mu(1+i\omega)}$
\ncross section
\n
$$
\times \int d^{3}x e^{-ix(\mathbf{p}-\mathbf{p}')/\hbar} K \left(\mathbf{x}, \mathbf{p}'; \frac{u\hbar}{p^{0} - E(\mathbf{p}')} \right). \qquad (16)
$$
\n
$$
\text{function which is}
$$
\n
$$
\times \int d^{3}x e^{-ix(\mathbf{p}-\mathbf{p}')/\hbar} K \left(\mathbf{x}, \mathbf{p}'; \frac{u\hbar}{p^{0} - E(\mathbf{p}')} \right). \qquad (16)
$$
\n
$$
\text{function which is}
$$
\n
$$
\begin{aligned}\n&\left(11\right) &\frac{\alpha}{\hbar} \left(\mathbf{x}, \
$$
$$

 Next, we consider the Yukawa potential, Next, we consider the Yukawa potential, $V(\mathbf{x})\!=\!\lambda e^{-kM|\mathbf{x}|}/\|\mathbf{x}\|$ where M is mass of a particle that mediating force. For this potential, we consider only Finally, the last exponential term of Eq. (18) by using approximation and incomplete Gamma function. We obtain $\int d^3 \mathbf{p} e$ and modialing force. The lane potential, we conclude the last exponential term of Eq. (18) by using approxima-
 $\int d^3\mathbf{n} d^3\mathbf{n}$ the last exponential term of Eq. (18) by using approxim
tion and incomplete Gamma function. We obtain $\frac{1}{2}$ tion and incomplete Gamma function. We obtain approximation and incomplete Gamma function. We rhat mediating force. For this potential, we consider only
the last exponential term of Eq. (18) by using approxima*ⁱ ^u e K* la incomplete Camma function. W Thus Eq. (13) becomes the comes of the comes of the computation of the that mediating force. For this potential, we con- $V(\mathbf{x}) = \lambda e^{-\kappa n |\mathbf{x}|} / |\mathbf{x}|$ where *M* is mass of a particle
that mediating force. For this potential, we consider only $V(\mathbf{x}) = \lambda e^{-kM|\mathbf{x}|}/|\mathbf{x}|$ where M is mass of a partic
that mediating force. For this potential, we consider on iast exponential term of Eq. (16) by u
and incomplete Gamma function. W tion and incomplete Gamma function. We obtain T_{NOM} **p** . (16) oonential term of Eq. (18) by using ap_l
complete Gamma function. We obtair riast exponential term of Eq. (16) by using approximation.
and incomplete Gamma function. We obtain the last exponential term of Eq. (18) by using ap
tion and incomplete Gamma function. We obtai

$$
\exp\left[-\frac{i}{\hbar}\int_0^{u\hbar/(p^0 - E(\mathbf{p}^*))}\mathrm{d}\alpha V\left(\mathbf{x} - \frac{\mathbf{p}^{\prime}}{m}\alpha\right)\right]
$$

$$
\approx \exp\left[-\frac{i}{\hbar}\frac{\lambda k m M}{|\mathbf{p}^{\prime}|}\left(\ln\left(\frac{2}{|\mathbf{x}|(1-\cos\theta)}\right) - \gamma\right)\right]
$$
(19)

where γ is Euler-Mascheroni constant and $\theta = \mathbf{p} \cdot \mathbf{x} / \|\mathbf{p}^\star\| \mathbf{x}$. Eq. (19) is independent of u $G^0_+(\mathbf{p})$ variable. Therefore, we can easily integrate over u in Eq. (18) and get ρ α **p** . (15) **p** . (15) **p** . (15) $\theta = \mathbf{p}' \cdot \mathbf{x} / ||\mathbf{p}'|| \mathbf{x}$ $||\mathbf{p}'||$. Eq. (19) is *m* **p** $\mathbf{p} \cdot \mathbf{x} / ||\mathbf{p}|| ||\mathbf{x}||$. Eq. (19) is independent of where γ is Euler-Mascheroni constant and $\rho^{i\beta\gamma}e^{-i\beta\ln(2p)}$ where γ is Eq.
 $\theta = \mathbf{p} \cdot \mathbf{x} / |\mathbf{p}'| |\mathbf{x}|$. Eq. $\| \mathbf{x} \|$. Eq. (1) where γ is Euler-Mascheronic:
 $\theta = \mathbf{p} \cdot \mathbf{x} / ||\mathbf{p}|| ||\mathbf{x}||$. Eq. (19) is indeper γ is Euler-M where γ is Euler-Mascheroni cor Eq. (10) and got where γ is Euler-Masch where γ is Euler-Mascheroni cor
 $\theta = \mathbf{n} \cdot \mathbf{x} / |\mathbf{n}'| |\mathbf{x}|$ Eq. (19) is independ vanable. Therefore, we can Euler-Mascherc where γ is Euler-Mascheroni constant a variable. Therefore, we can easily
Eq. (18) and get γ is Euler-Ma where γ is Euler-Mascheroni con

$$
\int_0^\infty \mathrm{d}u e^{iu(1+i\omega)} = \frac{-1}{i(1+i\omega)}\,. \tag{20}
$$

Finally, from Eq. (18), when $\Box \rightarrow +0$, we obtain Finally, from Eq. (18), when $\ln \rightarrow +0$, we o

$$
\int d^3 \mathbf{p} e^{i\mathbf{p}\cdot\mathbf{x}/\hbar} G_+(\mathbf{p}, \mathbf{p}'; p^0)
$$

\n
$$
\approx \frac{e^{i\mathbf{x}\cdot\mathbf{p}'/\hbar}}{[p^0 - E(\mathbf{p}') + i\omega]} e^{-i\beta \ln(2p')} e^{i\beta \gamma} e^{i\beta \ln(p'\mathbf{x} - \mathbf{p}'\cdot\mathbf{x})}. \quad (21)
$$

 (19) Green function for Yukawa scattering. In terms of energywhere $\beta = \lambda kmM / p'$. This is the asymptotically free
(19) Green function for Yukawa scattering. In terms of energyfree Green function for Yukawa scattering. In terms momentum representation, it is expressed as

$$
G_{+}^{0}(\mathbf{p}) = \frac{e^{i\beta\gamma}e^{-i\beta\ln(2p)}}{[p^{0} - E(\mathbf{p}) + i\omega]}.
$$
 (22)

1.5 MeV, as shown in Figure 1. u in
The meaning of "free" word is the **x**-independent. Therefore, we can plot this propagator for $M=0.510\,\,,\overline{1}\,$ and $\overline{1.5}$ MeV, as shown in Figure 1. 1.5 MeV, as shown in Figure 1.

Figure 1 The Asy Figure 1 The Asymptotically free Green function of the Yukawa scattering of various masses

The meaning of "free" word is the **x** -independent. $T_{\rm T}$ is can plot the plot therefore, we can plot the plot this propagator for $\int \langle \mathbf{p}, \mathbf{p} \rangle$ **M** α or β and β The scattering amplitude $f(\mathbf{p}, \mathbf{p}')$ for scattering The scattering amplitude $f(\mathbf{p}, \mathbf{p})$ for scattering particle word is the meaning of \mathbf{r} -independent. $T_{\rm eff}$ \mathcal{F}_max therefore, we can plot this propagator for \mathcal{F}_max The scattering amplitude *f* (, ') **p p** for scattering The scattering amplitude $f({\bf p},{\bf p}^{\, \prime})\;$ for scattering particle is given by

$$
f(\mathbf{p}, \mathbf{p}') = -\frac{m}{2\pi\hbar^2} \int d^3 \mathbf{p} \,^{\text{T}} V(\mathbf{p} - \mathbf{p}')
$$

$$
\times G_+(\mathbf{p}"', \mathbf{p}'; \, p^0) [p^0 - E(\mathbf{p}')] \Big|_{p^0 = E(\mathbf{p}')} , \tag{23}
$$

Discussion and Conclusion where $\, {\bf p} \,$ and ${\bf p} \, '$ are the initial and final momenta respectrajectory which involved potential *V* . This potential tively. Next, we substitute the final result from Eq.(21) and train the scattering amplitude as obtain the scattering amplitude as **resulting** trajectory which involved potential *V* . This potential where **p** and **p**' are the initial and final momenta respectively. vely. Next, we substitute the final result from Eq.(21) and **the substitual and** plant the scattering amplitude as

$$
f(\mathbf{p}, \mathbf{p'}) = -\frac{m}{2\pi\hbar^2} \frac{4\pi\lambda}{(\mathbf{p} - \mathbf{p'})^2 + (kM)^2}
$$

$$
\times (p'(\mathbf{p} - \mathbf{p'}) - \mathbf{p'} \cdot (\mathbf{p} - \mathbf{p'})^{i\beta} e^{i\beta\gamma} e^{-i\beta \ln(2\mathbf{p'})}. \quad (24)
$$

 After we take the squared absolute and then we get the differential cross section for Yukawa scattering. It is given as we get the differential cross section for \mathcal{G}

$$
D(\theta) = \frac{4m^2\lambda^2}{\hbar^4 \left(4{\bf p}^2\sin^2(\theta/2) + (kM)^2\right)^2}.
$$
 (25)

The result given in Eq. (25) is consistent with The result given in Eq. (25) is consistent with the Born approximation. By using Eq. (25), it can be shown by the graph in Figure 2, for $M = 0.510$,1 and 1.5 MeV, espectively. $F_{\rm F}$. The assessment of the $\frac{1}{2}$ the $\frac{1}{2}$ of $\frac{1}{2}$ of $\frac{1}{2}$

Discussion and Conclusion Discussion and Conclusion

The transformation function in Eq. (10) provided the trajectory which involved potential V . This potential is the function of $\mathbf{x} \!-\! \mathbf{p} (t \!-\! \tau)$ / m only, by setting $\, \mathbf{F} \! = \! 0 \,$

. So, this function leads to the propagator of scattered particle.The following result is the asymptotically free Green function for Yukawa scattering. It is directly obtained from considering this potential when a particle is near the energy shell. In Figure 1, the propagators still conserve their wave properties for the mass M of its behearte alon have proportion of the mass M or the . own potential for $M=1$ and 1.5 MeV respectively. Particularly, this means that the Yukawa potential decreased the influence of the incoming particle. It certainly confirms the existence of the short range potential. Finally, the differential cross section in Eq. (25) is obtained and consistent with the result calculated by the Born approximation method. Figure 2 shows the graph of the differential cross section for various masses as a comparative study by the present method.

Figure 2 The differential cross section of Yukawa scattering with various masses.

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References

- 1. Low, F. E. 2004. Classical Field Theory: Electromagnetism and Gravitation. Weinheim, Germany: WILEY-VCH Verlag GmbH & Co. KGaA.
- 2. Yukawa, H. 1934. On the Interaction of Elementary Particles I. Proceeding of the Physico-Mathematical Society of Japan. 17, 48-57.
- 3. Feynman, R. P. 1946. Space-Time Approach to Quantum Electrodynamics. Physics Review. 76, 769.
- 4. Feynman, R. P. 1946. The Theory of Positrons. Physics Reviews. 76, 749.
- 5. Schwinger, J. 1951. On the Green's Functions of Quantized Fields I. Proceedings of the National Academy of Sciences of the United State of America. 37, 452-455.
- 6. Schwinger, J. 1953. The Theory of Quantized Fields II. Physics Review. 91, 713-728.
- 7. Schwinger, J. 1960a. Unitary Transformations and the Action Principle. Proceedings of the National Academy of Sciences of the United State of America. 46, 883-897.
- 8. Schwinger, J. 1960b. The Special Canonical Group. Proceedings of the National Academy of Sciences of the United State of America. 46, 1401-1415.
- 9. Schwinger, J. 1961. Quantum Variables and the Action Principle. Proceedings of the National Academy of Sciences of the United State of America. 47, 1075-1083.
- 10. Schwinger, J. 1966. Particles and Sources. Physics review. 152, 1219-1226.
- 11. Manoukian, E. B. 1985. Quantum Action Principle and Path Integrals for Long-Range Interactions. Nuovo Cimento A. 90, 295-306.
- 12. Manoukian, E. B. 1988. Functional approach to scattering in quantum field theory. Inter International journal of theoretical physics. 27, 401-.
- 13. Manoukian, E. B. 2006. Quantum Theory: A wide spectrum. Netherlands: Springer.
- 14. Manoukian, E. B. and Sukkhasena, S. 2006. Functional Treatment of Quantum Scattering via the Dynamical Principle. Progress of Theoretical Physics. 116, 795-801.