การกระเจิงของศักย์ยูกาว่าโดยการใช้หลักการควอนตัมเชิงพลวัต

Yukawa scattering treated by the Quantum dynamical principle

อาณาจักร พลจันทึก, เสกสรร สุขะเสนา

Anajak Phonchantuek[1], Seckson Sukhasena[2]

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บทคัดย่อ

การกระเจิงโดยศักย์ยูกาว่า ถูกยกเป็นกรณีการศึกษา ผ่านการใช้เทคนิคการคำนวณโดยหลักการเชิงพลวัตควอนตัมที่เสนอโดยชวิงเงอร์ ซึ่งเป็นวิธีการที่ขึ้นกับฟังก์ชันกำเนิดที่ถูกแทนที่ด้วยตัวกระทำการแบบฟังก์ชันเชิงอนุพันธ์ จากผลลัพธ์เราได้ลักษณะอซิมโทติกของกรีนฟังก์ชันอิสระ ที่สามารถอธิบายลักษณะของการกระเจิงของอนุภาคต่อศักย์ยูกาวา และทำการแปรค่าพารามิเตอร์ของมวลให้มีค่าต่างๆ กัน นอกจากนี้ผลลัพธ์ที่ได้นี้ยังนำไปสู่ค่าแอมพลิจูดของการกระเจิงและค่าภาคตัดขวางของการกระเจิงเชิงอนุพันธ์อันเนื่องมาจากศักย์ยูกาว่าอีกด้วย

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Abstract

Yukawa scattering is pedagogically interpreted, by the Schwinger's quantum dynamical principle involving the generating function, which is replaced by a functional differential operation. As for the results, we get the asymptotically free Green function that explains the behavior of the Yukawa potential when the mass parameter is increasing and it can also lead to scattering amplitude and differential cross section respectively.

Keywords: quantum dynamical principle, Yukawa scattering, short range potentials, Green functions.

Introduction

In quantum scattering, we are interested in an interaction between the incident particles and the potential of the target e.g., coulomb potential[1] $V(x)=1/x$ which describes the behavior of particle scattering. Yukawa[2] presented his study by considering the meson interaction, particle with mass, which eventually was called the Yukawa potential,. Experimentally, researchers studied the scattering amplitude to determine these scattered particles. R. Feynman presented a diagram of particle scattering with the path integral that uses the time-slicing derivation[3,4].

Accordingly, in this report, we use the quantum dynamical principle proposed by J. Schwinger[5-9] to describe this situation. This method is very useful because it gives us the interested transformation function, also called the propagator. In particular, the Hamiltonian equation of this system involves external sources which generate degrees of freedom[10,11,12]. The equation is precisely derived from the variation of the transformation

[1] นิสิตระดับปริญญาโท วิทยาลัยเพื่อการค้นคว้าระดับรากฐาน "สถาบันสำนักเรียนท่าโพธิ์ฯ" มหาวิทยาลัยนเรศวร อำเภอเมือง จังหวัดพิษณุโลก 65000

[2] อาจารย์, วิทยาลัยเพื่อการค้นคว้าระดับรากฐาน "สถาบันสำนักเรียนท่าโพธิ์ฯ" มหาวิทยาลัยนเรศวร อำเภอเมือง จังหวัดพิษณุโลก 65000

[1] The Institute for Fundamental Study "The Tah Poe Academia Institute" Naresuan University, Phitsanulok, 65000, Thailand, anajakp58@email.nu.ac.th

[2] Lecturer, The Institute for Fundamental Study "The Tah Poe Academia Institute" Naresuan University, Phitsanulok, 65000, Thailand. Author for correspondence: secksons@nu.ac.th

function that depends on the potential, and then it is replaced by the differential functional. Consequently, it leads to the scattering amplitude.

The main purpose of this paper is to find the scattering amplitude and differential cross section by evaluating the asymptotically free Green function through the Yukawa potential. Previously, this method was also used to explain Coulomb scattering[13,14] near an energy shell. Clearly, this paper also shows the process, by setting tools, for interpreting the scattering problem in quantum theory by using the Yukawa potential which is involved with the mass term.

Quantum dynamical principle for scattering case

We start with a typical Hamiltonian written as

$$\mathrm{H}=\frac{\mathbf{p}^2}{2m}+V(\mathbf{x}), \tag{1}$$

where $\mathbf{p}$ is the momentum of a particle with mass m and incident on a potential $V(\mathbf{x})$. Furthermore, we present a new Hamiltonian $\mathrm{H}\ (\lambda,\tau)$ as follows

$$\mathrm{H}\ (\lambda,\tau)=\frac{\mathbf{p}^2}{2m}+\lambda V(\mathbf{x})-\mathbf{x}\cdot\mathbf{F}(\tau)+\mathbf{p}\cdot\mathbf{S}(\tau), \tag{2}$$

The latter involves the external sources $\mathbf{F}(\tau)$ and $\mathbf{S}(\tau)$ at time τ. These sources are linear function of $\mathbf{x}$ and $\mathbf{p}$. The sources generate $\mathbf{x}(\tau)$ and $\mathbf{p}(\tau)$, for position and momentum at time τ respectively. The parameter λ is an arbitrary parameter. The arbitrary parameter is another physical quantity involved with the system that we don't need to specify. It will be eventually set equal to one (because of the boundary condition of the transformation function).

Next we introduce Schwinger's quantum dynamical principle in the variation of transformation function from $\mathbf{p}$ at time t' (initial state) to $\mathbf{x}$ at time t (final state), written as

$$\delta\langle \mathbf{x}t\,|\,\mathbf{p}t'\rangle=-\frac{i}{\hbar}\int_{t'}^{t}\mathrm{d}\tau\langle \mathbf{x}t\,|\,\delta\mathrm{H}\ (\mathbf{x}(\tau),\mathbf{p}(\tau),\tau;\lambda)\,|\,\mathbf{p}t'\rangle. \tag{3}$$

This leads to

$$\delta\langle \mathbf{x}t\,|\,\mathbf{p}t'\rangle=-\frac{i}{\hbar}\int_{t'}^{t}\mathrm{d}\tau\delta\left[\lambda V\left(-i\hbar\frac{\delta}{\delta\mathbf{F}(\tau)}\right)\right]\langle \mathbf{x}t\,|\,\mathbf{p}t'\rangle, \tag{4}$$

by inserting the Hamiltonian from Eq. (2) into Eq. (3). So, this variation satisfies this Hamiltonian and depends on the parameter λ. In addition, for $V(\mathbf{x})$, $\mathbf{x}$ is replaced by $-i\hbar\delta/\delta\mathbf{F}(\tau)$, which was denoted earlier.

Immediately, integrating Eq. (4) over λ from $\lambda=0$ to 1, we obtain

$$\langle \mathbf{x}t\,|\,\mathbf{p}t'\rangle=\exp\left[-\frac{i}{\hbar}\int_{t'}^{t}\mathrm{d}\tau V\left(-i\hbar\frac{\delta}{\delta\mathbf{F}(\tau)}\right)\right]\langle \mathbf{x}t\,|\,\mathbf{p}t'\rangle^{0}\Big|_{\mathbf{F}=0,\mathbf{S}=0}, \tag{5}$$

where $\langle \mathbf{x}t\,|\,\mathbf{p}t'\rangle$ satisfied the Hamiltonian in Eq. (1) and setting the parameter $\lambda=1$. The transformation function $\langle \mathbf{x}t\,|\,\mathbf{p}t'\rangle^{0}$ is governed by the free Hamiltonian. The source terms, $\mathbf{F}(\tau)$ and $\mathbf{S}(\tau)$ are finally set equal to zero to satisfy the Hamiltonian in Eq. (1). Zero-superscript denotes the free particle where it's Hamiltonian is given by

$$\mathrm{H}\ (0,\tau)=\frac{\mathbf{p}^2}{2m}-\mathbf{x}\cdot\mathbf{F}(\tau)+\mathbf{p}\cdot\mathbf{S}(\tau). \tag{6}$$

By applying the quantum dynamical principle to Eq.(6) and replacing $\mathbf{p}$ by $i\hbar\delta/\delta\mathbf{S}(\tau)$, we obtain

$$\langle \mathbf{x}t\,|\,\mathbf{p}t'\rangle^{0}=\exp\left[-\frac{i}{2m\hbar}\int_{t'}^{t}\mathrm{d}\tau\left(i\hbar\frac{\delta}{\delta\mathbf{S}(\tau)}\right)^{2}\right]\langle \mathbf{x}t\,|\,\mathbf{p}t'\rangle_{0}. \tag{7}$$

The transformation function $\langle \mathbf{x}t\,|\,\mathbf{p}t'\rangle_{0}$ satisfies the Hamiltonian equation, i.e. without the kinetic term, which is defined as

$$H_0(\tau)=-\mathbf{x}\cdot\mathbf{F}(\tau)+\mathbf{p}\cdot\mathbf{S}(\tau). \tag{8}$$

By the way, the *Heisenberg equation* of the special Hamiltonian in Eq. (8) is integrated. Finally, we obtain

$$\langle \mathbf{x}t\,|\,\mathbf{p}t'\rangle_{0}=\exp\left[\frac{i}{\hbar}\mathbf{x}\cdot\left(\mathbf{p}+\int_{t'}^{t}\mathrm{d}\tau\mathbf{F}(\tau)\right)\right]$$

$$\times\exp\left[-\frac{i}{\hbar}\mathbf{p}\cdot\int_{t'}^{t}\mathrm{d}\tau\mathbf{S}(\tau)\right]$$

$$\times\exp\left[-\frac{i}{\hbar}\int_{t'}^{t}\mathrm{d}\tau\int_{t'}^{t}\mathrm{d}\tau'\mathbf{S}(\tau)\Theta(\tau-\tau')\mathbf{F}(\tau')\right], \quad (9)$$

where the Heaviside step function, $\Theta(\tau)$, is used as the time controller. In particular, from Eq. (9), when we set $\mathbf{S}(\tau)$ and $\mathbf{F}(\tau)$ equal to zero and substitute into Eq. (7) thus Eq. (5) is directly rewritten as

$$\langle \mathbf{x}t\,|\,\mathbf{p}t'\rangle=\exp\left[\frac{i}{\hbar}\left(\mathbf{x}\cdot\mathbf{p}-\frac{\mathbf{p}^2}{2m}(t-t')\right)\right]$$

$$\times\exp\left[\frac{i\hbar}{2m}\int_{t'}^{t}\mathrm{d}\tau\int_{t'}^{t}\mathrm{d}\tau'[t-\tau_>]\frac{\delta}{\delta\mathbf{F}(\tau)}\cdot\frac{\delta}{\delta\mathbf{F}(\tau')}\right]$$

$$\times\exp\left[-\frac{i}{\hbar}\int_{t'}^{t}\mathrm{d}\tau V\left(\mathbf{x}-\frac{\mathbf{p}}{m}(t-\tau)+\mathbf{F}(\tau)\right)\right]\Bigg|_{\mathbf{F}=0,\mathbf{S}=0}, \quad (10)$$

where $\tau_>$ is maximum of τ and τ'. We replace, for the potential function, $-i\hbar\delta/\delta\mathbf{F}(\tau)$ by $\mathbf{x}-\mathbf{p}(t-t')/m$ to get Eq. (10). Finally, we get the translational invariant in time when setting $\mathbf{F}(\tau)=0$.. $\langle \mathbf{x}t\,|\,\mathbf{p}t'\rangle$ is a *transformation function* of $t-t'$.

The differential cross section from the asymptotically free Green function

Next, we determine the differential cross section by using the *asymptotically free Green function* which is a function at infinite time.

We recall the definition of Green function that

$$\begin{aligned}\langle \mathbf{x}t\,|\,\mathbf{p}t'\rangle&=G_+(\mathbf{x}t,\mathbf{p}t')\\&=\int \mathrm{d}^3\vec{x}'e^{i\mathbf{x}'\cdot\mathbf{p}/\hbar}G_+(\mathbf{x}t,\mathbf{x}'t'),\end{aligned} \quad (11)$$

for $G_+(\mathbf{x}t,\mathbf{x}'t')$ is denoted as $\langle \mathbf{x}t\,|\,\mathbf{x}'t'\rangle$.

We use the Fourier transform to rewrite Eq. (11) as

$$\begin{aligned}G_+(\mathbf{p},\mathbf{p}';p^0)&=-\frac{i}{\hbar}\frac{1}{(2\pi\hbar)^3}\int_0^\infty \mathrm{d}\alpha e^{i(p^0+i\wp)\alpha/\hbar}\\&\times\int \mathrm{d}^3\mathbf{x}e^{-i\mathbf{x}\cdot\mathbf{p}/\hbar}\langle \mathbf{x}\alpha\,|\,\mathbf{p}'0\rangle,\end{aligned} \quad (12)$$

where $\langle \mathbf{x}\alpha\,|\,\mathbf{p}'0\rangle$ is given in Eq. (10) with $t-t'\equiv\alpha$, and $\wp\to+0$.

Next, insert Eq. (10) into Eq. (12), we obtain

$$G_+(\mathbf{p},\mathbf{p}';p^0)=-\frac{i}{\hbar}\frac{1}{(2\pi\hbar)^3}\int_0^\infty \mathrm{d}\alpha e^{i(p^0-E(\mathbf{p}')+i\wp)\alpha/\hbar}$$

$$\times\int \mathrm{d}^3\vec{x}e^{-i\mathbf{x}\cdot(\mathbf{p}-\mathbf{p}')/\hbar}K(x,\mathbf{p}';\alpha), \quad (13)$$

where $E(\mathbf{p}')=\mathbf{p}'^2/2m$.

The function $K(\mathbf{x},\mathbf{p}';\alpha)$ is denoted as

$$K(\mathbf{x},\mathbf{p}';\alpha)=\exp\left[\frac{i\hbar}{2m}\int_{t'}^{t}d\tau\int_{t'}^{t}d\tau'[t-\tau_>]\frac{\delta}{\delta\mathbf{F}(\tau)}\frac{\delta}{\delta\mathbf{F}(\tau')}\right]$$

$$\times\exp\left[-\frac{i}{\hbar}\int_{t'}^{t}\mathrm{d}\tau V\left(\mathbf{x}-\frac{\mathbf{p}'}{m}(t-\tau)+\mathbf{F}(\tau)\right)\right]\Bigg|_{\mathbf{F}=0}. \quad (14)$$

For $\alpha\to\infty$, we consider the $G_+(\mathbf{p},\mathbf{p}';p^0)$ near the energy shell $p^0\simeq\mathbf{p}'^2/2m$. So, we introduce the new integration variable as

$$u=\frac{\alpha}{\hbar}[p^0-E(\mathbf{p}')]. \quad (15)$$

Thus Eq. (13) becomes

$$G_+(\mathbf{p},\mathbf{p}';p^0)[p^0-E(\mathbf{p}')]=-\frac{i}{(2\pi\hbar)^3}\int_0^\infty \mathrm{d}u e^{iu(1+i\wp)}$$

$$\times\int \mathrm{d}^3\mathbf{x}e^{-i\mathbf{x}\cdot(\mathbf{p}-\mathbf{p}')/\hbar}K\left(\mathbf{x},\mathbf{p}';\frac{u\hbar}{p^0-E(\mathbf{p}')}\right). \quad (16)$$

Considering only $K(\mathbf{x},\mathbf{p}';u\hbar/p^0-E(\mathbf{p}'))$ where we set $\mathbf{F}(\tau)=0$, and obtain

$$K\left(\mathbf{x},\mathbf{p}';\frac{u\hbar}{p^0-E(\mathbf{p}')}\right)$$

$$\simeq\exp\left[-\frac{i}{\hbar}\int_0^{u\hbar/(p^0-E(\mathbf{p}'))}\mathrm{d}\alpha V\left(\mathbf{x}-\frac{\mathbf{p}'}{m}\alpha\right)\right] \quad (17)$$

Clearly, Eq. (16) can be rewritten as

$$\begin{aligned}&G_+(\mathbf{p},\mathbf{p}';p^0)[p^0-E(\mathbf{p}')]\\&=-\frac{i}{(2\pi\hbar)^3}\int_0^\infty \mathrm{d}u e^{iu(1+i\wp)}\int \mathrm{d}^3\mathbf{x}e^{-i\mathbf{x}\cdot(\mathbf{p}-\mathbf{p}')/\hbar}\end{aligned}$$

$$\times\exp\left[-\frac{i}{\hbar}\int_0^{u\hbar/(p^0-E(\mathbf{p}'))}\mathrm{d}\alpha V\left(\mathbf{x}-\frac{\mathbf{p}'}{m}\alpha\right)\right]. \quad (18)$$

Next, we consider the Yukawa potential, $V(\mathbf{x})=\lambda e^{-kM|\mathbf{x}|}/|\mathbf{x}|$ where M is mass of a particle that mediating force. For this potential, we consider only the last exponential term of Eq. (18) by using approximation and incomplete Gamma function. We obtain

$$\exp\left[-\frac{i}{\hbar}\int_0^{u\hbar/(p^0-E(\mathbf{p}'))}\mathrm{d}\alpha V\left(\mathbf{x}-\frac{\mathbf{p}'}{m}\alpha\right)\right]$$

$$\simeq\exp\left[-\frac{i}{\hbar}\frac{\lambda kmM}{|\mathbf{p}'|}\left(\ln\left(\frac{2}{|\mathbf{x}|(1-\cos\theta)}\right)-\gamma\right)\right] \quad (19)$$

where γ is Euler-Mascheroni constant and $\theta=\mathbf{p}'\cdot\mathbf{x}/|\mathbf{p}'||\mathbf{x}|$. Eq. (19) is independent of u variable. Therefore, we can easily integrate over u in Eq. (18) and get

$$\int_0^{\infty}\mathrm{d}u e^{iu(1+i\epsilon)}=\frac{-1}{\mathrm{i}(1+i\epsilon)}. \quad (20)$$

Finally, from Eq. (18), when $\epsilon\rightarrow+0$, we obtain

$$\int\mathrm{d}^3\mathbf{p}e^{i\mathbf{p}\cdot\mathbf{x}/\hbar}G_+(\mathbf{p},\mathbf{p}';p^0)$$

$$\simeq\frac{e^{i\mathbf{x}\cdot\mathbf{p}'/\hbar}}{[p^0-E(\mathbf{p}')+i\epsilon]}e^{-i\beta\ln(2p')}e^{i\beta\gamma}e^{i\beta\ln(p'x-\mathbf{p}'\cdot\mathbf{x})}. \quad (21)$$

where $\beta=\lambda kmM/p'$. This is the asymptotically free Green function for Yukawa scattering. In terms of energy-momentum representation, it is expressed as

$$G_+^0(\mathbf{p})=\frac{e^{i\beta\gamma}e^{-i\beta\ln(2p)}}{[p^0-E(\mathbf{p})+i\epsilon]}. \quad (22)$$

The meaning of "free" word is the $\mathbf{x}$-independent. Therefore, we can plot this propagator for $M=0.510$, 1 and 1.5 MeV, as shown in Figure 1.

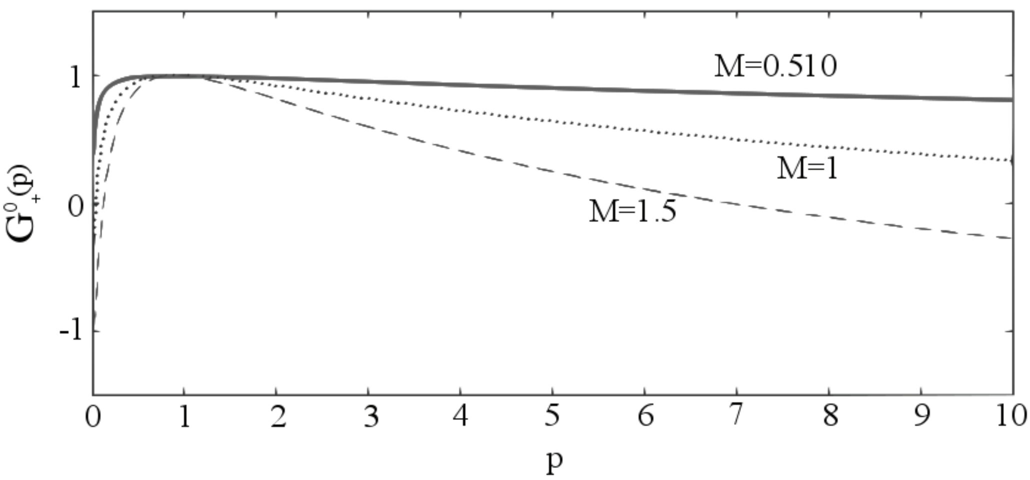

Figure 1 The Asymptotically free Green function of the Yukawa scattering of various masses

The scattering amplitude $f(\mathbf{p},\mathbf{p}')$ for scattering particle is given by

$$f(\mathbf{p},\mathbf{p}')=-\frac{m}{2\pi\hbar^2}\int\mathrm{d}^3\mathbf{p}''V(\mathbf{p}-\mathbf{p}')$$

$$\times G_+(\mathbf{p}'',\mathbf{p}';p^0)[p^0-E(\mathbf{p}')]\Big|_{p^0=E(\mathbf{p}')}, \quad (23)$$

where $\mathbf{p}$ and $\mathbf{p}'$ are the initial and final momenta respectively. Next, we substitute the final result from Eq.(21) and obtain the scattering amplitude as

$$f(\mathbf{p},\mathbf{p}')=-\frac{m}{2\pi\hbar^2}\frac{4\pi\lambda}{(\mathbf{p}-\mathbf{p}')^2+(kM)^2}$$

$$\times\left(p'(p-p')-\mathbf{p}'\cdot(\mathbf{p}-\mathbf{p}')\right)^{i\beta}e^{i\beta\gamma}e^{-i\beta\ln(2p')}. \quad (24)$$

After we take the squared absolute and then we get the differential cross section for Yukawa scattering. It is given as

$$D(\theta)=\frac{4m^2\lambda^2}{\hbar^4\left(4\mathbf{p}'^2\sin^2(\theta/2)+(kM)^2\right)^2}. \quad (25)$$

The result given in Eq. (25) is consistent with the Born approximation. By using Eq. (25), it can be shown by the graph in Figure 2, for $M=0.510$,1 and 1.5 MeV, respectively.

Discussion and Conclusion

The transformation function in Eq. (10) provided the trajectory which involved potential V . This potential is the function of $\mathbf{x}-\mathbf{p}(t-\tau)/m$ only, by setting $\mathbf{F}=0$. So, this function leads to the propagator of scattered particle.The following result is the asymptotically free Green function for Yukawa scattering. It is directly obtained from considering this potential when a particle is near the energy shell. In Figure 1, the propagators still conserve their wave properties for the mass M of its own potential for $M=1$ and 1.5 MeV respectively. Particularly, this means that the Yukawa potential decreased the influence of the incoming particle. It certainly confirms the existence of the short range potential. Finally, the differential cross section in Eq. (25) is obtained and consistent with the result calculated by the Born approximation method. Figure 2 shows the graph of the differential cross section for various masses as a comparative study by the present method.

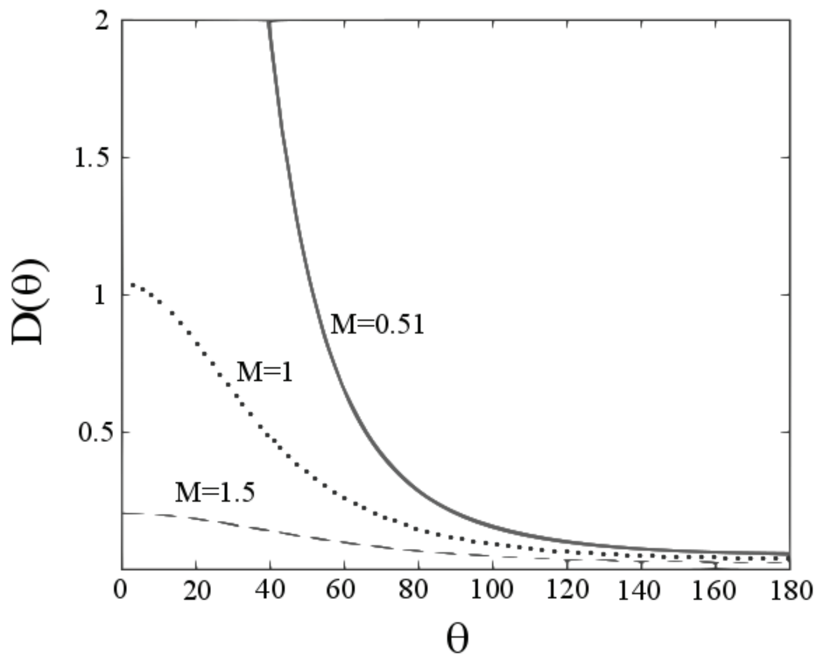

Figure 2 The differential cross section of Yukawa scattering with various masses.

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