อัตราสวนของลําดับยอยของจํานวนฟโบนักชีที่มีดัชนีเปนเลขชี้กําลัง *ⁿ*

The ratio of the \emph{n} -th exponential subsequence of the Fibonacci Sequence

วิภาวี ตั้งใจ $^{\text{!}*}$, กรภัทร ชมฮด $^{\text{2}}$ Wipawee Tangjai^{1*}, Korrapat Chomhod² Received: 12 March 2018; Accepted: 22 May 2018 the recurrence relationship

บทคัดยอ

เป็นที่ทราบกันดีว่าอัตราส่วนของพจน์ที่ติดกันของจำนวนฟีโบนักชี $\{F_m\}_{m=0}^\infty$ และอัตราส่วนของพจน์ที่ติดกันของจำนวนลูคัส ี ลู่เข้าสู่อัตราส่วนทองคำ งานวิจัยนี้ศึกษาลำดับย่อย { F_{m} า}ື = เมื่อ n เป็นจำนวนเต็มบวก โดยได้แสดงว่าลิมิตของ อัตราส่วนระหว่าง $\frac{F(m+1)^n}{n}$ และ $\frac{F_m n}{n}$ ลู่เข้าก็ต่อเมื่อ $n\leq 2$ โดยทำการพิสูจน์ลำดับที่เกิดจากความสัมพันธ์เวียนเกิดอันดับสอง ในรูปทั่วไปที่ครอบคลุมลําดับฟโบนักชี นอกจากนี้ยังไดใหคาของลิมิตที่เกิดขึ้น $\overline{\hat{D}}$ ซี *{F*...}∞_。และอัตราส่วนของพจน์ที่ติดกันของจำนวนลคัส พิสจน์ลำดับที่เกิดจา $\frac{d}{dt}$ ତାଙ୍କ recurrence relationships $\frac{d}{dt}$ 1 2 , *a ka a m mm* for *m* 2 (2) where *a s* ⁰ and ¹ *a t* . We assume that *FF F mm m* 1 2 **,** for *m* 2, 1 าลิมิตที่เกิดขึ้น
ลิมิตที่เกิดขึ้น F_{m} ก $\}_{m=0}^{\infty}$ เมื่อ n เป็นจำนวนเต็มบวก โดยได้แสดงว่าลิมิตของ δ กัส เลคัส $\;$ if *kst* 1, 0, 1, then *an* is the เวนฟีโบนักชี $\{F_m\}_{m=0}^\infty$ และอัตราส่วนของพจน์ที่ติดกันของจำนวนลูคัส r $<$ 2 โดยทำการพิสลน์ลำดับที่เกิดจากคาาบสับพัน ก์ให้ค่าของลิมิตที่เกิดขึ้น Fibonacci number , *Fn* segments and its relation to a homogeneous

คำสำคัญ: ลำดับฟีโบนักชี อัตราส่วน การลู่เข้า ความสัมพันธ์เวียนเกิด ลำดับย่อยที่มีดัชนีเป็นเลขชี้กำลังเป็น $\,n\,$ $\tilde{\pi}$ มพันธ์เวียนเกิด ลำดับย่อยที่มีดัชนีเป็นเลขชี้กำลังเป็น $\,n\,$

Abstract

It is well known that the ratios of the consecutive terms of the Fibonacci numbers $\{F_m\}_{m=0}^\infty$ and those of the Lucas numbers $\{L_m\}_{m=0}^\infty$ converge to the golden ratio. In this work, we study the n -exponential subsequence $\{F_{m^n}\}$, where *n* is a positive integer. We show that the limit of the quotient between $\frac{F_{(m+1)^n}}{F_{(m+1)^n}}$ and $\frac{F_m}{F_{(m+1)^n}}$ converges if and only if by proving a more general statement for the sequences satisfying a recurrence relation of order 2 that covers the Fibonacci sequence. We also give the limit of the convergence if it exists. Fibonacci sequence. We also give the limit of the convergence if it ex non-zero real number *^k*, we let *^m*⁰ *^m ^a* non-zero real number *^k*, we let *^m*⁰ *^m ^a* if *kst* 1, 2, 1, then *an* is the if *kst* 2, 0, 1, then *an* is the achees salisfying a recarrence is
convergence if it exists. if *kst* 2, 2, 2, then *an* is the uda
. if *kst* 2, 2, 2, then *an* is the terms of and if **k** $\frac{1}{2}$ where Pell-Lucas number *Qn* . $hat covers$ i. *x Mxy y* (,) , for all 0 *x y* ii. *M x y Mxy* (,) (,), for

Keyword: Fibonacci sequence, Quotient, Convergence, Recurrence relation, *n* -exponential subsequence **Introduction**
 Internal Electronical Securities Conversance Becurence relation in expensatiol subsequent The *Fibonacci sequence ^m ^m* ⁰ *^F* Recurrence relation, n -exponential subsequence The *Fibonacci sequence ^m ^m* ⁰ *^F* all, , (0,) *x y* . all, , (0,) *x y* .

Introduction the recurrence relationship of $\mathbf r$

The *Fibonacci sequence* $\left\{F_{_m}\right\}_{m=0}^{\infty}$ *is defined by the* recurrence relation ${\mathcal{F}}$ *for* ${\mathcal{F}}$ \mathcal{F} \mathcal{F} \mathcal{F} is defined by

$$
F_m = F_{m-1} + F_{m-2}, \text{ for } m \ge 2,
$$
 (1)

where $F_0 = 0$ and $F_1 = 1$. In 2015, Craciun¹ defined a geometrical generalization of the golden ratio ratio by considering a ratio between two sub-ratio by considering a ratio between two sub-ratioby considering a ratio between two sub-relation to a homogeneous function *M* defined by function *M* defined by by considering a ratio between two sub-segments and its se a geometrie generation of and generation ion to a homogeneous

$$
M\!:\!(0,\infty)\!\times\!(0,\infty)\!\rightarrow\!(0,\infty)
$$

function *M* defined by satisfying satisfying satisfying satisfying

and

satisfying

i.
$$
x < M(x, y) < y
$$
, for all $0 < x < y$

i. *x Mxy y* (,), for all 0 *x y*

 $\frac{1}{2}$ \mathbb{R} *M* $(\lambda x, \lambda y) = \lambda M(x, y)$ $\frac{1}{2} M(2n, 2n) - 3M(n, n)$ for all ii. $M(\lambda x, \lambda y) = \lambda M(x, y)$, for all $\lambda, x, y \in (0, \infty)$. ii. $M(\lambda x, \lambda y) = \lambda M(x, y)$, for all $\lambda, x, y \in (0, \infty)$ $u(x, y)$, for all $\lambda, x, y \in (0, \infty)$.

 $\frac{1}{\sqrt{2\pi}}$ $\frac{1}{\sqrt{2\pi}}$ recorrections are least relations in the recurrence relations in the recurrence relations in the recurrence relations of $\frac{1}{\sqrt{2\pi}}$ $\frac{3}{2}$ ways, one of which is the k – Fibonacci by, for a non-zero integer k , 1 **1 2 , and multiple is have been grown**
2 , and multiple is the motor multiple is the motor multiple is the motor multiple is the motor multiple in the m The Fibonacci numbers have been generalized The Fibonacci numbers have been generalized in many ways, one of which is the $k -$ Fibonacci numbers² defined ∞ . many ways, one of which is the *k* Fibonacci positive integer, then $\mathbf{3}$ positive integer, then \mathcal{I} (1) by, for a non-zero integer k , $\frac{1}{2}$ positive integer, then $\frac{1}{2}$ ned and the set of the
Set of the set of the s , ,1 ,2 , *F kF F km km km* for 2, *m* where ,0 *Fk* = 0 and ,1= 1. *Fk* It is well known

 $F_{k,m} = kF_{k,m-1} + k$ $F_{k,m} = kF_{k,m-1} + F_{k,m-2}$, for $m \ge 2$,

if *kst* 2, 2, 2, then *an* is the

In this paper, we are interested in the growth

 $T_{k,0} = 0$ and $F_{k,1} = 1$. It is $\begin{aligned} \n\text{SFR}^1 \text{ where } F_{k,0} &= 0 \text{ and } F_{k,1} = 1. \n\end{aligned}$ to the golden ratio $\varphi = \frac{1+\sqrt{5}}{2}$. If we con-
expense is the expression $\int_{\mathcal{F}}^{\infty}$ if φ *Fibonaccial* subseque sequence, it is obvious that the ratio of conse
goes to infinity. goes to infinity. aciun¹ where $F_{k,0} = 0$ and $F_{k,1} = 1$. It is well known
aciun¹ where $F_{k,0} = 0$ and $F_{k,1} = 1$. It is well known fhe ratio of consecutiv to the golden ratio $\alpha = \frac{1+\sqrt{5}}{2}$ If we const Lucas number , *Ln* Lucas number , *Ln* $\binom{m}{m}$ _{*m*=1}
sequence it is obvious that the ratio of consec goes to infinity.
We will stree ne ratio of consecutive Fibonacci num
— ψ golden ratio $\varphi = \frac{1}{2}$. **If we consider** onential subsequence $\left\{ F_{_{m^{n}}}\right\} _{m=1}^{\infty}$ of the Fibonacci sequence, it is obvious that the ratio of consecutive term
goes to infinity. where *p* is a positive integer and 2 4 . ² *^k* $t_{\rm max}$ that the $t_{\rm max}$ where numbers converges
F $t = \frac{1}{\pi}$ if we consider the n where the consideration subsequence subseq where $T_{k,0} = 0$ and $T_{k,1} = 1$. *First* we define the same of annoncing Γ is more and Γ that the ratio of consecutive Fibonacci numbers $1+\sqrt{5}$ or consecutive $1+\sqrt{5}$ $\gamma = 2$
-exponential subsequence $\left\{F_{m^n}\right\}_{m=1}^{\infty}$ of the Fibonacci where *p* is a positive integer and 2 4 . ² *^k* t the limit of the t where *p* is a positive integer and 2 4 . ² *^k* \mathbf{S} ند.
That the ratio of consecutive Fibonacci numbers converges ا and its to the golden ratio $\varphi = \frac{1+\sqrt{5}}{2}$. If we consider the *n* goes to infinity. converges if and only if *n* 2, and sequence, it is obvious that the ratio of consecutive terms

We will study a more generalized form of the Fibonacci and K **if then are in the state of the state o** Fibonacci and k – Fibonacci numbers. For a non-zero \mathbf{r} that if *n* 2, then the limit converges to ² *^k* . that if α 2 α 2 α 2 α . α 2 α Fibonacci and *k* − Fibonacci numbers. For a non-zero

a

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the Fibonacci and *k* Fibonacci numbers. For a

Considering a more generalized sequence

¹ภาควิชาคณิตศาสตร คณะวิทยาศาสตร มหาวิทยาลัยมหาสารคาม, อําเภอกันทรวิชัย จังหวัดมหาสารคาม 44150 i. *x Mxy y* (,) , for all 0 *x y* ณิตศาสตร์ คณะวิทยาศาสตร์ มหาวิทยาลัยมหาสารคาม, อำเภอกันทรวิชัย จังหวัดมหาสารคา i. *x Mxy y* (,) , for all 0 *x y* ณิตศาสตร์ คณะวิทยาศาสตร์ มหาวิทยาลัยมหาสารคาม, อำเภอกันทรวิชัย จังหวัดมหาสารคาม 44150

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. author: Wipawee Tar

real number $\,k,\,$ we let $\left\{a_{\scriptscriptstyle m}\right\}_{\scriptscriptstyle m=0}^{\infty}$ be a sequence generated by a recurrence relation α , we let $\{a_m\}_{m=0}$ be a sequence where *a s* ⁰ and ¹ *a t* . We assume that real number $k,$ we let $\left\{a_{\scriptscriptstyle m}\right\}_{\scriptscriptstyle m=0}^\infty$ be a sequ *f* real number k , we let $\left\{a_m\right\}_{m=0}^\infty$ be a sequence generated $\qquad |k|$ 0, *ⁿ ma* for all *m n*, 1. **Main Theorem**

$$
a_m = ka_{m-1} + a_{m-2}, \text{ for } m \ge 2
$$

$$
(2)
$$
 Mai

$$
(2)
$$
 Mai

where $a_0 = s$ and $a_1 = t$. We assume that $a_{m^n} \neq 0$, for all $m, n \geq 1$. where $a - s$ and $a = t$ We satisfying (2): where $a - s$ and $a = t$ We assumed where $a_0 = s$ and $a_1 = t$. We assume that

satisfying (2): m_{m^n} by for an $m, n = 1$.
The followings are examples of the sequences satisfying (2): if *kst* 1, 0, 1, then *an* is the Lucas number , *Ln* **Theorem 2.1.**

- if $k = 1$, $s = 0$, $t = 1$, then a_n is the Fibonacci number F_n , \ldots \ldots $\int_{\mathcal{A}} f(x) dx$ is the *M* is the *M* is the *M* is the *M* is the *if* $\int_{\mathcal{A}} f(x) dx$ if $\int_{\mathcal{A}} f(x) dx$ is the *a*nd *i* $\int_{\mathcal{A}} f(x) dx$ is the *a*¹ is the *a* Lucas number , *Ln* • if $k = 1$, $s = 0$, $t = 1$, then a_n is the Fibona
- if $k = 1$, $s = 2$, $t = 1$, then a_n is the Lucas number L_n , if $k = 1$, $s = 2$, $t = 1$, then a_n \cdot **ii** κ - **1**, s - 2, ι - **1**, \mathbf{r} $\mathbf{$
- if $k = 2$, $s = 0$, $t = 1$, then a_n is the Pell number P_n , $\frac{1}{2}$, $\frac{2}{3}$, $\frac{1}{2}$, Pell number , *Pn* L_n ,
• if $k = 2$, $s = 0$ P_{α} , \sum_{n} $k = 2$, $s = 0$, $t = 1$, then a_n is the r en • if $k = 2$, $s = 0$, $t = 1$, then a_n is the Pell number $a_{m+1}^{(n)}$ \mathbf{F}_x , P_n , Pell-Lucas number *Qn* . P_n ,
- if $k = 2$, $s = 2$, $t = 2$, then a_n is the Pell-Lucas number Q_n . In $k = 2$, $k = 2$, and in *n*,

• if $k = 2$, $s = 2$, $t = 2$, then a_n is the Pell-Lucas $a_{(m+1)^n}$, $a_{(m+1)^n}$

In this paper, we are interested in the growth rate of such ratios which is the quotient of $\frac{a_{m^n}}{a_{m^n}}$. It has been shown that if k is a positive i rate of such ratios which is the quotient of $\frac{a_{(m+1)^n}}{a}$ and a_{m-1}^2
n $en^{3,4}$ $\frac{a_{m^n}}{a_n}$. It has rate of such ratios which is the quotient of ¹ $\frac{a_{m^n}}{a_{m^n}}$. It has been shown that if k is a positive integer, $\lim_{n \to \infty} \frac{(m-1)}{3}$ *m a a* $\frac{a_{m^{\prime \prime }}}{a}$. It has been shown that if k is a positive In this paper, we are interested in the growth
rate of such ratios which is the quotient of $\frac{a_{m+1}^m}{a}$ and *a* is been shown that if $\,k\,$ is a positive *a* In this paper, we are interested . It has been shown that if *k* is a $\frac{a_{m^n}}{a_{m-1)_i^n}}$. It has been shown that if k is a positive integer, $\frac{a^2 r_1^{\min\{m+n\}}}{a_{m-1)_i^n}}$ \overline{a} l in . It has been shown that if *k* is a *^k ^m k m*

$$
\lim_{m\to\infty}\frac{F_{k,m+p}}{F_{k,m}}\ =\ \varphi_k^p\ ,
$$
 (3)

where p where p is a positive integer and where p is a positive integer and where p is a positive integer and

 $\omega_a = \frac{k + \sqrt{k^2 + 4}}{2}$, By (3), it can limit of the quotient of $\frac{F_{k,(m+1)^n}}{F_{k,m^n}}$ and $\frac{1}{I}$ $\frac{1}{\sqrt{2}}$ *<i>Phistering a more* d only if $n \leq 2$, and that if $n = 2$, sequence $\left\{a_{m}\right\}_{m=0}$ we give a result rel a_{m^n} $a_{(m-1)^n}$ $k + \sqrt{k^2 + 4}$ $\varphi_k = \frac{k + \sqrt{k^2 + 4}}{2}$. By (3), it can be verified that
limit of the quotient of $\frac{F_{k,(m+1)^n}}{F_{k,m}}$ and $\frac{F_{k,m^n}}{F_{k,m}}$ conver that if $n = 2$, then limit converges to φ_k^2 . Considering a more nce ∤
∴ sequence $\{a_i\}^\infty$ we give a result related to the quot $\frac{a}{a}$ and $\frac{a}{a}$ in Considering a more generalized sequence limit of the quotient of $\frac{R_{k,m^n}}{F_{k,m^n}}$ and $F_{k,(m-1)}^{'}$ $a_{(m+1)^n}$ a_{m^n} of $\frac{m}{a_{m^n}}$ and $\frac{m}{a_{m^{n-1}}}$ in $\varphi_k = \frac{k + \sqrt{k^2 + 4}}{2}$. By (3), it can be verified mit of the quotient of $\frac{x,(m+1)}{F}$ and $\frac{x,n}{F}$ *n k m F* $\overline{F_{k,(m-1)^n}}$ co if and only if $n \leq 2$, and that if $n = 2$, then the *FIIL* CC it converges to ω^2 . Considering a more gen the equence $\{a_m\}_{m=0}^{\infty}$ we give a result related to the $\varphi_k = \frac{k + \sqrt{k^2 + 4}}{2}$. By (3), it can be verified imit of the quotient of $\frac{R_{k,m^n}}{F_{k,m^n}}$ and $\frac{R_{k,n^n}}{F_{k,(m^n)}}$ $k,(m-1)^n$ rges to $\,\phi^{\scriptscriptstyle z}_{\hspace{-1.2pt}k}$. Considering a more ger a_{m+1}^n a_{m}^n a_{m^n} and $a_{(m-1)^n}$ By (3), it can be verified that the limit of the quotient of $\omega = \frac{k + \sqrt{k^2 + 4}}{k}$ By (3), it can limit of the quotient of $\frac{k,(m+1)}{E}$, *n n k m k m F* $\frac{k,(m+1)^n}{F_{k-m^n}}$ and converges if and only if *n* 2, and sequence $\left\{a_m\right\}_{m=0}^\infty$ we give a result related to the quotient $\left(\alpha r_1^{m^m}-\beta\right)^{m-1}$ of $\frac{u_{m+1}^n}{a}$ and $\frac{u_{m}^n}{a}$ in where p is a positive
 $k + \sqrt{k^2 + 4}$ $\varphi_k = \frac{\varphi_k}{2}$. By (3), fifted the quotient of F_{k,m^n} limit converges to φ_k^2 . Considering a more generalized $=\frac{\alpha^2 r_1^{m_1m_2m_3m_4m_5} + \beta^2 r_1^{m_1m_2m_4m_5} + \$ $(m-1)$ $\frac{k,m^n}{(m-1)^n}$ *k m k m F* $F_{k,(m-1)}^{k, m-1}$ if and only if $n \leq 2$, and that if $n = 2$, then the $\varphi_k = \frac{K + \sqrt{K}}{2}$ *n n k m k m F* $\frac{K + \sqrt{K} + 4}{2}$. and $\frac{a}{a}$ of $\frac{a_{(m+1)^n}}{a_{m^n}}$ and $\frac{a_{m^n}}{a_{(m-1)^n}}$ in *n m m a* $\frac{\left(m+1\right)^{n}}{a_{m^{n}}}$ and $(m-1)$ *n n m m a* $a_{\binom{m}{2}}$ f $\frac{a_{(m+1)^n}}{a}$ and $\frac{a_{m^n}}{a}$ i *m m a* $\frac{m+1)^n}{u_{m^n}}$ and $(m-1)$ *n n m m a* $a_{\binom{m}{2}}$ of $\frac{(m+1)^n}{a}$ and $\frac{a_{m^n}}{a}$ in *m* of $\frac{(m+1)}{a_{m^n}}$ *m a* of $\frac{a(n+1)^n}{a}$ and $\frac{a_m}{a}$ *n m m a* $\frac{(m+1)^n}{a_{m^n}}$ and $(m$ *n n m m a* $a_{\binom{m}{2}}$ of $\frac{(m+1)}{a}$ and $\frac{m^m}{a}$ *n* $\frac{a_{\binom{m}{2}}}{a_{\binom{m}{2}}}$ *m* $\frac{a^{(m+1)}}{a_{m^n}}$ and $\frac{n}{a_{(m+n)}}$ 1 *m a m* $a_{\binom{m}{2}}$ in
I *m m a* $\frac{\sqrt{n+1}}{2}$. $1,(m+1)$ *n n m m a a*
k,(m $, |$ In $n \geq 2$, and that if $n-2$, then $\sum_{i=1}^{n} a_{i}$ $\binom{m}{i}$ $\binom{n}{i}$ $\frac{1}{a}$ and $\frac{1}{a}$ in $a_{(m-1)^n}$

Theorem 2.1.

In 2016, R. Euler and J. Sadek⁵ showed that 1 2 *r* r .
D . Eulen and 1. Cadal⁵ ab sured that

$$
a_m = \frac{1}{r_1 - r_2} \Big(\alpha r_1^m - \beta r_2^m \Big), \tag{4}
$$

where $\alpha = s - tr_2$, $\beta = t - sr_1$ and

$$
r_1, r_2 \in \left\{ \frac{k + \sqrt{k^2 + 4}}{2}, \frac{k - \sqrt{k^2 + 4}}{2} \right\}
$$
 such that

 $|r_{\textrm{i}}|~> |r_{\textrm{i}}|~$. We note that $~0~<~|r_{\textrm{i}}|~<~1.$

Main Theorem Main Theorem Main Theorem

Considering $\{a_n\}_{n=0}^{\infty}$ is **Considering** ℓ \mathbf{r} $\frac{n-3}{2}$ δ onsidering $\{a^{\dagger}\}$ $\frac{1}{2}$ Considering $\{a_n\}_{n=0}^{\infty}$ satisfying (2),

we let $\sum_{n \geq n}$

We assume that
$$
a_m^{(n)} = \frac{a_{m^n}}{a_{(m-1)^n}}.
$$

Theorem 2.1. **Theorem 2.1. 2.1.**

is the Fibonacci
\n
$$
\lim_{m \to \infty} \frac{a_{m+1}^{(n)}}{a_m^{(n)}} = \begin{cases} 0, & \text{if } n = 1, \\ r_1^2, & \text{if } n = 2, \\ \infty, & \text{otherwise.} \end{cases}
$$
\nthe Lucas number

 $\frac{1}{2}$ cases names.
Proof. By using the Binet formula of a_m in (4), u_m in (4),

ⁿ is the Pell number
$$
\frac{a_{m+1}^{(n)}}{a_m^{(n)}}
$$

\nⁿ is the Pell-Lucas $\frac{a_{(m+1)^n}}{a_{m^n}} \cdot \frac{a_{(m+1)^n}}{a_{m^n}}$
\n $\frac{a_{(m+1)^n}}{a_{m^n}}$ and $\frac{a_{(m+1)^{n+1}}^{(m+1)^n} - a_{(m+1)^n}^{(m+1)^n} - a_{(m+1)^n}^{(m+1)^n} - a_{(m+1)^n}^{(m+1)^n}}{a_{m^n}^{(m-n)} - a_{m^n}^{(m-n)} - a_{m^n}^{(m+n)} - a_{m^n}^{$

$$
+\frac{\beta^2 r_2^{(m-1)^n+(m-1)^n}-\alpha\beta\bigl(-1\bigr)^{(m-1)^n} \,r_2^{(m+1)^n-(m-1)^n}}{\Bigl(\alpha r_1^{m^n}-\beta r_2^{m^n}\Bigr)^2}\\
$$

Vol 37. No 5, September-October 2018 The ratio of the *n*-th exponential subsequence of
\n
$$
= \frac{\alpha^2 r_1^{(m+1)^n + (m-1)^n - 2m^n} - \alpha \beta (-1)^{(m-1)^n} r_1^{(m+1)^n - (m-1)^n - 2m^n}}{\left(\alpha - \beta \left(\frac{r_1}{r_1}\right)^{m^n}\right)^2} \qquad l_1 = \frac{k_1 + \sqrt{k_1^2 + 4k_2}}{2}
$$
\n
$$
+ \frac{\beta^2 r_2^{(m+1)^n + (m-1)^n - 2m^n} - \alpha \beta (-1)^{(m-1)^n} r_2^{(m+1)^n - (m-1)^n - 2m^n}}{\left(\alpha \left(\frac{r_1}{r_2}\right)^{m^n} - \beta\right)^2} \qquad \text{and}
$$
\n
$$
l_2 = \frac{k_1 - \sqrt{k_1^2 + 4k_2}}{2}.
$$
\nIf $1 - k_1 < k_2 < 0$, then 0
\nWe have
\n
$$
\lim_{m \to \infty} \frac{\alpha^2 r_1^{(m+1)^n + (m-1)^n - 2m^n} - \alpha \beta (-1)^{(m-1)^n} r_1^{(m+1)^n - (m-1)^n - 2m^n}}{\left(\alpha - \beta \left(\frac{r_2}{r_1}\right)^{m^n}\right)^2} \qquad \text{Theorem 2.2. If } b_m, \text{ is no}
$$
\n
$$
l - k_1 < k_2 < 0, \text{ in (6), the}
$$
\n
$$
\left(\alpha - \beta \left(\frac{r_2}{r_1}\right)^{m^n}\right)^2 \qquad l_1 < k_2 < 0, \text{ in (6), the}
$$
\n
$$
\frac{b_m^{(m)}}{b_m^{(m)}} = \begin{cases} 0, & \text{if } n = 1, \\ l_1^2, & \text{if } n = 2, \\ \infty, & \text{if } n = 3. \end{cases}
$$
\nAs a result, the quotient of
\ntial subsequence of the Fit
\nto the square of the golden
\nLet F_m , L_m , P_m , Q_m be

We have

We have
\n
$$
\lim_{m\to\infty} \frac{\alpha^2 r_1^{(m+1)^n + (m-1)^n - 2m^n} - \alpha \beta (-1)^{(m-1)^n} r_1^{(m+1)^n - (m-1)^n - 2m^2}}{\left(\alpha - \beta \left(\frac{r_2}{r_1}\right)^{m^n}\right)^2}
$$
\n
$$
= \begin{cases}\n0, & \text{if } n = 1, \\
r_1^2, & \text{if } n = 2, \\
\infty, & \text{if } n = 3.\n\end{cases}
$$
\n
$$
= \begin{cases}\n0, & \text{if } n = 1, \\
r_1^2, & \text{if } n = 2, \\
\infty, & \text{if } n = 3.\n\end{cases}
$$
\n
$$
= \begin{cases}\n0, & \text{if } n = 1, \\
r_1^2, & \text{if } n = 2, \\
\infty, & \text{if } n = 3.\n\end{cases}
$$
\n
$$
= \begin{cases}\n0, & \text{if } n = 1, \\
r_1^2, & \text{if } n = 2, \\
\infty, & \text{otherwise.} \\
\text{Since } 0 < |r_2| < 1 \text{ and } |r_2| < |r_1|, \text{, it follows that} \\
\text{Let } F_1, F_2, F_1, P_1, Q_2, \text{ be a } 0.\n\end{cases}
$$

Since $0 < |r_2| < 1$ and $|r_2| < |r_1|$, it follows that 1 and $|r_2|$ Since $0 < |r_2| < 1$ and $|r_2| < |r_1|$, it follow $0 < |r_2| < 1$ and $|r_2| < |r_1|$, , it follows tha $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$ implies that $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$ implies that ≥ 1 and $|r| \geq |r|$ it follows that $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$, the ratios of the

$$
\lim_{m \to \infty} \frac{\beta^2 r_2^{(m+1)^n + (m-1)^n - 2m^n} - \alpha \beta (-1)^{(m-1)^n} r_2^{(m+1)^n - (m-1)^n - 2m^2}}{\left(\alpha \left(\frac{r_1}{r_2}\right)^{m^n} - \beta\right)^2} = 0.
$$

Therefore, \mathbf{r} *n* By (5), we can also conclude that consecutive terms of the *n*exponential subsequence ⁰ *nm ^m ^a n* 2.

Therefore,
\n
$$
\frac{a_{m+1}^{(n)}}{a_m^{(n)}} = \begin{cases}\n0, & \text{if } n = 1, \\
r_1^2, & \text{if } n = 2, \\
\infty, & \text{otherwise.} \n\end{cases}
$$

n n nn n n

 \mathbf{S} , \mathbf{S} , we can also conclude that \mathbf{S} By (5), we can also conclude that $a_{(m+1)^{(n)}}$ By (5), we can also conclude that a_{ℓ} (*s*) By (5), we can also cond
 $\frac{a}{b}$ we can also co By (5), we can also conclude that $\frac{a}{b}$ *k* are can also conclude that

$$
\frac{a_{(m+1)^{(n)}}}{a_m^{(n)}} \in O(\mathbf{r}_1^{2m^{n-2}}).
$$

Theorem 2.1 implies that, for any positive integer k , the \mathbb{C}^{2n-1}_{m-1} growth rate of the ratios of consecutive terms of the $n -$ Corollary 2.4. $\frac{1}{\sqrt{2}}$ converse terms of the $\frac{1}{\sqrt{2}}$ converse. subsequence $\left\{ a_{m^n} \right\}_{m=0}$ converges $\left\{ a_{m^n} \right\}_{m=0}$ \mathbf{L} $\frac{1}{2}$ consecutive terms of the *nexponential* subsequence $\left\{a\right\}^{\infty}$ converges if and $\frac{1}{2}$ only if $n \leq 2$. α . grown rate of the ratios of consecutive terms of the $n = 1$ **Cord** $\sum_{m=0}$ $E[\text{S} \mid \mathcal{L}]$, rue exponential subsequence $\left\{ a_{m^n} \right\}_{m=0}^{\infty}$ converges if and $\left\{ k + \sqrt{k^2 + 4} \right\}$ nteger κ , the $\frac{1}{2}$ **Corollary 2.4.** $\left(\frac{1}{2} - \frac{1}{2}\right)^2$ Theorem 2.1 implies that for any positive in κ , the sponential subsequence $\{u\}$ I heorem 2.1 implies that, for any positive inte growth rate of the ratios of consecutive terms
example the ratios of the terms of the security characteristic experience $\{a_{m^n}\}_{m=0}$ converted. ϵ characteristic equation of ϵ_n are distinct formula of ϵ are distinct for ϵ Theorem 2.1 implies that, for any positive int $\left\{ a_{m^n} \right\}_{m=0}$ consequence $\left\{ a_{m^n} \right\}_{m=0}$ consequence uence $\{a_{n}\}^{\infty}$ converg \int_{α}^{∞} and the construction of the state of $\lim_{m \to 0}$ converges if a \geq 2.

Theorem 2.1 can be generalized to the sequences $T_{\rm eff}$ can be generalized to the generalized to ${b_m}_{m=0}^{\infty}$ defined by Theorem 2.1 can be generalized to the Theorem 2.1 can be generalized to the sequences $\lim_{x \to 0^+} \frac{F_{k,m}^{(2)}}{F_{k,m}^{(2)}} =$ Theorem 2.1 can be generali:
 $\left\{ b_{m}\right\} _{m=0}^{\infty}$ defined by y if $n \le 2$.
eorem 2.1 can be generalized to the sequences $\lim_{m \to \infty} \frac{F_{k,m}^{(2)}}{F^{(2)}} = \begin{cases} \frac{1}{2} & \text{if } k = 1, 2 \end{cases}$ Theorem 2.1 can be generalized to the
 $\{b_m\}_{m=0}^{\infty}$

$$
b_m = k_1 b_{m-1} + k_2 b_{m-2}, \text{ for } m \ge 2
$$
 (6)
where k k are non-negative integers and

where $k_{\scriptscriptstyle 1}, k_{\scriptscriptstyle 2}$ are non-negative in ω_0 ω , ω_1 \cdots in the reduce of the original $\frac{1}{\sqrt{2}}$ and $\frac{1}{\sqrt{2}}$ are non-negative integers and $\frac{1}{\sqrt{2}}$ are distinct. then the Binet $b_0 = s$, $b_1 = t$. If the roots of the characteristic 2.4.

Solution of (6) are distinct than the Binet formula of h is equation of (6) are distinct, then the Binet formula of b_m is where n_1, n_2 are non-negative where k_1, k_2 are non-negative integers and
 $k_1 = s$, $k_2 = t$, if the roots of the characteristic are non-negative $b_1 = t$. It where K_1, K_2 are non-negative integers and k_1 , $b_1 = t$. If the roots of the characteri 11 2 $\overline{1}$ \overline{a} .
t_› are non-negative i

$$
b_m = \frac{1}{l_1 - l_2} \Big((t - s l_2) l_1^m + (s l_1 - t) l_2^m \Big),
$$
\nEXAMPLE Xample 2.5. Let $U_m - U_m$ is a constant. The equation is given by the formula U_m and U_m is a constant. The equation is given by U_m and U_m are the values of U_m and U_m are the values of U_m . The equation is given by U_m and U_m are the values of U_m and U_m are the values of U_m and U_m . The equation is the equation of U_m and U_m are the values of U_m and U_m . The equation is the equation of U_m and U_m are the values of

where

$$
l_1 = \frac{k_1 + \sqrt{k_1^2 + 4k_2}}{2}
$$

and

 $\qquad \qquad \lbrack \infty ,$

$$
l_2 = \frac{k_1 - \sqrt{k_1^2 + 4k_2}}{2} \ .
$$

If $1-k_1 < k_2 < 0$, then $0 < l_2 < 0$ If $1-k_1 < k_2 < 0$, then $0 < l_2 < 1$ and $l_2 < l_1$.

 (5) So, we are able to extend (5) So, we are able to extend the same method appearing
in Theorem 2.1 to Theorem 2.2. in Theorem 2.1 to Theorem 2.2. *m m* So, we are able to extend the same method app
in Theorem 2.1 to Theorem 2.2.
Theorem 2.2. If b_{m^n} is not zero for all $m, n \ge 1$ $\frac{1}{2}$. n Theorem 2.1 to Theorem 2.2. *n* able to extend the same
 n 2.1 to Theorem 2.2.
 2.2. If b_{m^n} is not zero for

 $(m+1)^n - (m-1)^n - 2m^2$ **Theorem 2.2.** If b_{m^n} is not zero for all $m, n \ge 1$ and **1−** $k_1 < k_2 < 0$, in (6), then (n) (n) $\frac{1}{\lambda} = \left\{ l_1^2 \right\}$ 0, *if* $n=1$, \int if $n=2$, , otherwise. *n m n* $\frac{b_{m+1}^{(n)}}{b_m^{(n)}} = \begin{cases} 0, & \text{if } n = 1, \\ l_1^2, & \text{if } n = 2, \\ \infty, & \text{otherwise} \end{cases}$ $\begin{cases} \n\frac{1}{n!} \\ \n\frac{1}{n!} = \n\begin{cases} \n1 & \text{if } n = 0 \\ \n\frac{1}{n!} & \text{if } n = 0 \n\end{cases} \n\end{cases}$ α $\begin{bmatrix} 0, & 0 \end{bmatrix}$ *i* $- k_1 < k_2 < 0$,
*b*_{*on+1}</sup></sup> =* $\begin{cases} 0, \\ l_1^2, \end{cases}$ *</sub>* $\frac{1}{2}$ $b_n^{(n)} \quad \begin{array}{|c|} \hline 0, \end{array}$, 2, *if n* $\frac{b_{m+1}^{(n)}}{b_{n}^{(n)}} = \begin{cases} 0, & \text{if } n = 1, \\ l_1^2, & \text{if } n = 2, \end{cases}$ $m, n \ge 1$ and **Theorem 2.2.** If b_{m^n} is *in* Theorem 2.1 to Theore
Theorem 2.2. If b_{m^n} is
 $1 - k_1 < k_2 < 0$, in (6), then i Theorem 2.1 to The
heorem 2.2. If $b_{\scriptscriptstyle m^{\scriptscriptstyle n}}$ $b_m^{\gamma\gamma}$ ∞ otherwise. is not zero for 0, if $n=1$, $t_{n+1}^{n+1} = \begin{cases} l_1^2, & \text{if } n = 2, \end{cases}$ ∞ , otherwise.

Theorem 2.2. If *ⁿ mb* is not zero for all *m n*, 1 tial subsequence of the Fibonacci sequences converges $\frac{1}{2}$ As a result, the quotient of the ratios of As a result, the quotient of the ratios of the *n* − exponenthat to the square of the golden ratio. As a result, the quotient of the ratios of the $n-$ exponensequences to the square of the golden Let \mathbf{F}^{H} **Figure 1.** Figure 1. **Figure 1.** Figure 1. **Figure 1. Figure 1. Fi** an subsequence of the Fibonacci sequences of

 $\frac{-2m^2}{m} = 0.$ Number, Pell number and Pell-Lucas tively. Let φ be the golden ratio and $\delta =$
Corollary 2.3. The following statements a Let F_m , L_m , P_m , Q_m be the Fibonacci number, Lu *i* if the golden ratio and $\partial =$
the following statements are e golden ratio and δ
following statements tively. Let φ be the golden ratio and $\delta = 1 + \sqrt{2}$.
Corollary 2.3. The following statements are true:
 $F^{(2)}$ \mathcal{L}_m , \mathcal{L}_m be an number and Ft *n m* $\frac{m^2 - (m-1)^n - 2m^2}{m} = 0.$ Number, Pell number and Pell-Lucas number, respec- F_m , L_m , P_m , Q_m be the mber, Pell number and P Let F_m , L_m , P_m , Q_m be the Fibonacci number, Lucas

m Corollary 2.3. Th Corollary 2.3. The follow **Corollary 2.3.** The following statements are true:
 $F⁽²⁾$

Corollary 2.3. The following statements are true:
\n•
$$
\lim_{m \to \infty} \frac{F_{m}^{(2)}}{F_{m-1}^{(2)}} = \varphi^2
$$
\n•
$$
\lim_{m \to \infty} \frac{L_{m}^{(2)}}{L_{m-1}^{(2)}} = \varphi^2
$$
\n•
$$
\lim_{m \to \infty} \frac{P_{m}^{(2)}}{P_{m-1}^{(2)}} = \delta^2
$$
\n•
$$
\lim_{m \to \infty} \frac{Q_{m}^{(2)}}{Q_{m-1}^{(2)}} = \delta^2.
$$

\nitive integer *k*, the

 $\overline{}$ Corollary 2.4. \sim 2 4 \sim 2 \sim r *Q* . **Corollary 2.4.**

nverges if and
\nhe sequences
$$
\lim_{m \to \infty} \frac{F_{k,m}^{(2)}}{F_{k,m-1}^{(2)}} = \begin{cases} \left(k + \sqrt{k^2 + 4}\right)^2 & \text{if } k > 0, \\ \left(k - \sqrt{k^2 + 4}\right)^2 & \text{if } k < 0. \end{cases}
$$
\n(6)

 α P Example 2.5 gives an e rem 2.2 but not the seque lim lim *m Q* $\overline{}$ Example 2.5 g т.
См. на (6)
Example 2.5 gives an example of the sequences satisfying
Norse and nampio
le sequ *P* orem 2.2 but not the sequences lim $¹$ </sup> Tried 2 lim *m P* $\frac{1}{2}$ $\overline{2}$ $\begin{array}{ll} \text{tic} & 2.4. \ \end{array}$ **Example 2.5.** Let $b_{\text{w}} = 3b_{\text{w-1}} +$ Example 2.5 gives an example of the sequences satisfying
rs and
Theorem 2.2 but not the sequences in Corollary 2.3 and
teristic 2.4. $2.4.$ \mathbf{z} . \mathbf{z} but not the sequence in \mathbf{z} Example 2.5 gives an example of the sequences Example 2.5 gives an example of the sequences satisfying

 $0, b_1 = 1...$ Let 2 lim *m m Q* $\overline{1}$ $b_1 = 1...$ Table \cdot 1 *m P* $\ddot{}$ \mathbb{R}^2 $b_0 = 0$, *m Q* $b_1 = 1$. mple 2.5. Let $b_m = 3b_{m-1} + 2b_{m-1}$ 2 $b_0 = 0,$ *Q* $\overline{1}$ $\sqrt{2}$ $= 1...$ Table 1 show $h = 0$ $h = 1$ Table 1 show Exam *Q* Example 2.5 2 nple 2.<mark>!</mark> $\overline{}$ and $b_0 = 0, b_1 = 1$. Table 1 shows the set the sequence in th formula of $b_m^{}$ is $\qquad \qquad \textbf{Example 2.5.}$ Let $b_m^{} = 3b_{m-1}^{} + 2b_{m-2}^{}$, , where $b_0=0, b_1=1...$ Table 1 shows the value of b_{m^2} , for ...
. 1 **Example 2.5.** Let $b_m = 3b_{m-1} + 2b_{m-2}$, , where $\frac{1}{2}$ for $\frac{1}{2}$ $\sum_{i=1}^n$

$$
m = 1,...,10.
$$
 By Theorem 2.2,

$$
\lim_{m \to \infty} \frac{b_{(m+1)^2}b_{(m-1)^2}}{b_{m^2}} = \frac{\left(3 + \sqrt{17}\right)^2}{4}.
$$

Table 1: b_m^2

${\bf Discussion}$

Discussion T_{max} in T_{max} is the all sequences satisfying T_{max} $t - \kappa_1 < \kappa_2 < 0$, and ω_{m^n} is non-number $m, n \ge 1$. The growth of the ratios of computed real numbers, $\binom{n}{n}$, $\binom{n}{n}$ of the subsequence $\{b_{m^n}\}\$ is Theorem 2.2 implies that all sequences satisfying the recurrence relation (6) with a condition that $1-k_1 < k_2 < 0$, and b_{m^n} is non-null real number, for $m, n \ge 1$. The growth of the ratios of consecutive terms numbers. Chaos, Solitons and Fractals. 2007;32(5):1615 -1624. Available from: \mathbf{s} 2007;32(5):1615 -1624. Available from: com/science/article/pii/S0960077906008332 $\frac{1}{\sqrt{5}}$ Euler R, Sadek J. A direct proof that $\frac{1}{\sqrt{5}}$

 $C(t_1)$ it converges if and only if if $n = 2$, then *ⁿ ^m O l* [5] Euler R, Sadek J. A direct proof that *Fn* $O(l_1^{2m^{n-2}})$. It converges if and only if $n \leq 2$. Moreover, $\mathsf{ver},$

$$
\lim_{m\to\infty}\frac{b_{(m+1)^2}b_{(m-1)^2}}{b_{m^2}}=\frac{2k_1^2+2\sqrt{k_1^2+4k_2}+4k_2}{4}.
$$

2 $A = \frac{1}{2}$ A cknowledgment

2 4 *^m ^m b* This project is financially supported by the 2017 research funding of the Faculty of Science, Mahasarakham University, Thailand. We would also like to thank the anonymous reviewers for carefully read the paper and for the comments.

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