ลำดับใหม่ที่สอดคล้องกับลำดับ *k*-ฟีโบนักชี

Some novel sequences related to k-Fibonacci sequences

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บทคัดย่อ

งานวิจัยนี้เราได้นำเสนอสามลำดับรูปแบบใหม่ของ γ_n, α_n และ β_n ที่มีส่วนเกี่ยวข้องกันผ่านความสัมพันธ์เวียนเกิด และเราได้สังเกตถึงความสัมพันธ์ของลำดับทั้งสามนี้สามารถแสดงให้อยู่ในรูปของลำดับ *k*-ฟีโบนักชี เพื่อพิสูจน์ความสัมพันธ์นี้เราได้นำหลักอุปนัยเชิงคณิตศาสตร์ มาใช้สำหรับแสดงความถูกต้องของทฤษฎี และแสดงผลลัพธ์ที่ได้จากการศึกษาในงานนี้

คำสำคัญ: ลำดับ *k*-ฟีโบนักชี, ความสัมพันธ์เวียนเกิด, อุปนัยเชิงคณิตศาสตร์

Abstract

In this research, we introduce three novel sequences of γ_n, α_n and β_n. These sequences are related to each other through the recurrence relation, and we have observed that their relationship can be expressed using *k*-Fibonacci sequences. To prove this relationship, we used mathematical induction. We have shown the validity of our theorem, and the results are presented in this study.

Keywords: *k*-Fibonacci sequences, recurrence relation, mathematical induction

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Introduction

For any integer number $k \geq 1$, the n th k-Fibonacci sequence, denoted as $\{F_{k\text{-}n}\}^{\infty}_{n=0}$, is defined by (Falcon & Plaza, 2007) as a recursive sequence as follows:

$$F_{k,n+1} = kF_{k,n} + F_{k,n-1}$$

where $F_{k,0} = 0$ and $F_{k,1} = 1$. The first 8 members of k-Fibonacci sequences are shown below:

$0, 1, k, k^2 +1,..., k^3 + 2k, k^4 + 3k^2 + 1, k^5 + 4k^3 + 3k, k^6 + 5k^4 + 6k^2 + 1.$

(Atanassov, 2018) studied two new combined 3-Fibonacci sequences. Let a, b, c, d be arbitrary real numbers and $\{F_n\}^{\infty}_{n=0}$ be the standard Fibonacci sequence. The first set of sequences has the form for $n \geq 0$,

$$\alpha_{n+2} = \gamma_{n+1} + \beta_{n+1},$$
$$\beta_{n+2} = \gamma_{n+1} + \alpha_{n+1},$$
$$\gamma_{n+2} = \gamma_{n+1} + \gamma_{n}.$$

where $\alpha_0 = a, \beta_0 = b, \gamma_0 = c, \gamma_1 = d$. From these sequences and for each natural number $n \geq 1$ the result are the following,

$$\alpha_{2n+1} = b + F_{2n-1}a + (F_{2n}\text{-}1)d,$$
$$\alpha_{2n} = a + F_{2n}c + (F_{2n+1}\text{-}1)d,$$
$$\beta_{2n-1} = a + F_{2n-1}c + (F_{2n}+1)d,$$
$$\beta_{2n} = b + F_{2n}c + (F_{2n+1}\text{-}1)d,$$
$$\gamma_{n+2} = F_{n+1}c + F_{n+2}d.$$

The second set of sequences has the form for $n \geq 0$,

$$\alpha_{n+1} = \alpha_{n+1} + \alpha_{n},$$
$$\beta_{n+1} = \alpha_{n+1} + \gamma_{n},$$
$$\gamma_{n+1} = \alpha_{n+1} + \beta_{n}.$$

where $\alpha_0 = a, \beta_0 = b, \gamma_0 = c, \alpha_1 = d$. From these sequences and for each natural number $n \geq 1$ the result are the following,

$$\alpha_{n} = F_{n-1}c + F_{n}d,$$
$$\beta_{2n-1} = (F_{2n}\text{-}1)a + b + (F_{2n+1}\text{-}1)d,$$
$$\beta_{2n} = (F_{2n+1}\text{-}1)a + c + (F_{2n+2}\text{-}1)d,$$
$$\gamma_{2n-1} = (F_{2n}\text{-}1)a + c + (F_{2n+1}\text{-}1)d,$$
$$\gamma_{2n} = (F_{2n+1}\text{-}1)a + b + (F_{2n+2}\text{-}1)d.$$

In the same year, he studied two additional new combined 3-Fibonacci sequences part 2. Let a, b, c be arbitrary real numbers and $\{F_n\}^{\infty}_{n=0}$ be the standard Fibonacci sequence. The first set of sequences has the form for $n \geq 0$,

$$\alpha_{n+1} = \beta_{n} + \gamma_{n},$$
$$\beta_{n+1} = \alpha_{n} + \gamma_{n},$$
$$\gamma_{n+1} = \frac{\alpha_{n+1} + \beta_{n+1}}{2} + \gamma_{n},$$

where $\alpha_0 = 2a, \beta_0 = 2b, \gamma_0 = c$. From these sequences and for each natural number $n \geq 1$ the result are the following.

$$\alpha_{n} = (F_{2n-1}+(\text{-}1)^n)a + (F_{2n-1}\text{-}(\text{-}1)^n)b + F_{2n}c,$$
$$\beta_{n} = (F_{2n-1}\text{-}(\text{-}1)^n)a + (F_{2n-1}+(\text{-}1)^n)b + F_{2n}c,$$
$$\gamma_{n} = F_{2n}a + F_{2n}b + F_{2n+1}c.$$

The second set of sequences has the form for $n \geq 0$,

$$\alpha_{n+1} = \alpha_{n} + \frac{\beta_{n} + \gamma_{n}}{2},$$
$$\beta_{n+1} = \alpha_{n+1} + \gamma_{n},$$
$$\gamma_{n+1} = \alpha_{n+1} + \beta_{n}.$$

where $\alpha_0 = a, \beta_0 = 2b, \gamma_0 = 2c$. From these sequences and for each natural number $n \geq 1$ the result are the following.

$$\alpha_{n} = F_{2n-1}a + F_{2n}b + F_{2n}c,$$
$$\beta_{n} = F_{2n}a + (F_{2n+1} + (\text{-}1)^n)b + (F_{2n+1} \text{-} (\text{-}1)^n)c,$$
$$\gamma_{n} = F_{2n}a + (F_{2n+1} \text{-} (\text{-}1)^n)b + (F_{2n+1} + (\text{-}1)^n)c.$$

(Nubpetchploy & Pakapongpun, 2021) generated three combined sequences related to Jacobsthal sequences. Let a, b, c, d be arbitrary real numbers and J_n be the Jacobethal sequences. The first set of sequences has the form for $n \geq 0$,

$$\gamma_{n+2} = \gamma_{n+1} + 2\gamma_{n},$$
$$\alpha_{n+1} = \gamma_{n+1} + 2\beta_{n},$$
$$\beta_{n+1} = \gamma_{n+1} + 2\alpha_{n}.$$

where $\alpha_0 = a, \beta_0 = b, \gamma_0 = c, \gamma_1 = d$. From these sequences and for each natural number $n \geq 1$ the result are the following.

$$\gamma_n = 2J_{n-1}c + J_n d,$$
$$\alpha_n = 2\alpha_{n-1} + (J_n + (-1)^n)c + j_n d + (-2)^n(a-b),$$
$$\beta_n = 2\beta_{n-1} + (J_n + (-1)^n)c + j_n d - (-2)^n(a-b).$$

The second set of sequences has the form for $n \geq 0$,

$$\gamma_{n+2} = \gamma_{n+1} + 2\gamma_n,$$
$$\alpha_{n+1} = \gamma_n + 2\beta_n,$$
$$\beta_{n+1} = \gamma_n + 2\alpha_n.$$

where $\alpha_0 = a, \beta_0 = b, \gamma_0 = c, \gamma_1 = d$. From these sequences and for each natural number $n \geq 1$ the result are the following.

$$\gamma_n = 2J_{n-1}c + J_n d,$$
$$\alpha_n = 2\alpha_{n-1} + (J_{n-1} + (-1)^{n-1})c + j_{n-1}d + (-2)^n(a-b),$$
$$\beta_n = 2\beta_{n-1} + (J_{n-1} + (-1)^{n-1})c + j_{n-1}d - (-2)^n(a-b).$$

The third set of sequences has the form for $n \geq 0$,

$$\gamma_{n+1} = \frac{\alpha_{n+1} + \beta_{n+1}}{2} + 2\gamma_n,$$
$$\alpha_{n+1} = \gamma_n + 2\beta_n,$$
$$\beta_{n+1} = \gamma_n + 2\alpha_n.$$

where $\alpha_0 = 2a, \beta_0 = 2b, \gamma_0 = c$. From these sequences and for each natural number $n \geq 1$ the result are the following.

$$\gamma_{n-1} = (J_{2n-1}-1)(a+b) + J_{2n-1}c,$$
$$\alpha_n = (J^2_{n+1} - J^2_n+1)(a+b) + (-1)^n J_n a + (-1)^{n+1}(2J_{n+1} + J_n)b+J_{2n}c,$$
$$\beta_n = (J^2_{n+1} - J^2_n+1)(a+b) + (-1)^n J_n b + (-1)^{n+1}(2J_{n+1} + J_n)a+J_{2n}c.$$

(Atanassov, 2022) introduce on two new combined 3-Fibonacci sequences. Let a, b, c, d, e be arbitrary real numbers and $\{F_n\}^{\infty}_{n=0}$ be the standard Fibonacci sequence. The first set of sequences has the form for $n \geq 1$,

$$\alpha_{n+1} = \alpha_n + \alpha_{n-1},$$
$$\beta_{n+1} = \beta_n + \beta_{n-1},$$
$$\gamma_{n+1} = \frac{\alpha_n + \beta_n}{2} + \gamma_n.$$

where $\alpha_0 = 2a, \beta_0 = 2b, \gamma_0 = c, \alpha_1 = 2d, \beta_1 = 2e$. From these sequences and for each natural number $n \geq 1$ the result are the following.

$$\alpha_n = 2F_{n-1}a + 2F_n d,$$
$$\beta_n = 2F_{n-1}b + 2F_n e,$$
$$\gamma_n = F_n a + F_n b + c + (F_{n+1}-1)d + (F_{n+1}-1)e.$$

The second set of sequences has the form for $n \geq 1$,

$$\alpha_{n+1} = \alpha_n + \alpha_{n-1},$$
$$\beta_{n+1} = \beta_n + \beta_{n-1},$$
$$\gamma_{n+1} = \frac{\alpha_{n+1} + \beta_{n+1}}{2} + \gamma_n.$$

where $\alpha_0 = a, \beta_0 = b, \gamma_0 = c, \alpha_1 = 2d, \beta_1 = 2e$. From these sequences and for each natural number $n \geq 1$ the result are the following.

$$\alpha_n = 2F_{n-1}a + 2F_n d,$$
$$\beta_n = 2F_{n-1}b + 2F_n e,$$
$$\gamma_n = F_n a + F_n b + c + (F_{n+1}-1)d + (F_{n+1}-1)e.$$

(Pakapongpun & Kongson, 2022) introduced three combined sequences related to *k*-Fibonacci sequences. Let a, b, c, d be arbitrary real numbers and $\{F_{k,n}\}^{\infty}_{n=0}$ be the *k*-Fibonacci sequence. The first set of sequences has the form for $n \geq 0$,

$$\gamma_{n+2} = k\gamma_{n+1} + \gamma_n,$$
$$\alpha_{n+1} = k\gamma_n + \beta_n,$$
$$\beta_{n+1} = k\gamma_n + \alpha_n.$$

where $\alpha_0 = a, \beta_0 = b, \gamma_0 = c, \gamma_1 = d$. From these sequences the result are the following theorem 1.1.

Theorem 1.1. For any positive integer k and n,

(a) $\gamma_n = F_{k,n}d + F_{k,n-1}c,$

(b) $\alpha_{2n} = (F_{k,2n} + F_{k,2n-1} - 1)d + (F_{k,2n-1} + F_{k,2n-2} + (F_{k,2} - 1)c + a,$

(c) $\beta_{2n} = (F_{k,2n} + F_{k,2n-1} - 1)d + (F_{k,2n-1} + F_{k,2n-2} + (F_{k,2} - 1)c + b,$

(d) $\alpha_{2n-1} = (F_{k,2n-1} + F_{k,2n-2} - 1)d + (F_{k,2n-2} + F_{k,2n-3} + (F_{k,2} - 1)c + b,$

(e) $\beta_{2n-1} = (F_{k,2n-1} + F_{k,2n-2} - 1)d + (F_{k,2n-2} + F_{k,2n-3} + (F_{k,2} - 1)c + a.$

The second set of sequences has the form for $n \geq 0$,

$$\gamma_{n+2} = k\gamma_{n+1} + \gamma_n,$$
$$\alpha_{n+1} = k\gamma_{n+1} + \beta_n,$$
$$\beta_{n+1} = k\gamma_{n+1} + \alpha_n.$$

where $\alpha_0 = a, \beta_0 = b, \gamma_0 = c, \gamma_1 = d$. From these sequences the result are the following theorem 1.2.

Theorem 1.2. For any positive integer k and n,

(a) $\gamma_n = F_{k,n}d + F_{k,n-1}c,$

(b) $\alpha_{2n} = (F_{k,2n+1} + F_{k,2n} - 1)d + (F_{k,2n} + F_{k,2n-1} - 1)c + a,$

(c) $\beta_{2n} = (F_{k,2n+1} + F_{k,2n} - 1)d + (F_{k,2n} + F_{k,2n-1} - 1)c + b,$

(d) $\alpha_{2n-1} = (F_{k,2n} + F_{k,2n-1} - 1)d + (F_{k,2n-1} + F_{k,2n-2} - 1)c + b,$

(e) $\beta_{2n-1} = (F_{k,2n} + F_{k,2n-1} - 1)d + (F_{k,2n-2} + F_{k,2n-3} + (F_{k,2} - 1)c + a.$

The third set of sequences has the form for $n \geq 0$,

$$\gamma_{n+1} = k\gamma_n + \frac{\alpha_n + \beta_n}{2}$$
$$\alpha_{n+1} = k\gamma_n + \beta_n,$$
$$\beta_{n+1} = k\gamma_n + \alpha_n.$$

where $\alpha_0 = 2a, \beta_0 = 2b, \gamma_0 = c$. From these sequences, the result are the following theorem 1.3.

Theorem 1.3. For any positive integer k and n,

(a) $\gamma_{n+1} = \gamma_n(F_{k,2} + F_{k-1}) = \gamma_1(F_{k,2} + F_{k-1})^n,$

(b) $\alpha_{2n} = \gamma_1(F_{k,2} + F_{k-1})^{2n-1} + a - b,$

(c) $\alpha_{2n-1} = \gamma_1(F_{k,2} + F_{k-1})^{2n-2} + b - a.$

In this paper, we introduce a new three set of combined sequences which are more general context related to *k*-Fibonacci sequences.

Main Results

We applied those three sets of sequences from (Pakapongpun & Kongson, 2022) work as follows. Let a, b, c, d and s be arbitrary real numbers with $s \neq 0$. The first set of sequences has the form for $n \geq 0$,

$$\gamma_{n+2} = k\gamma_{n+1} + \gamma_n,$$
$$\alpha_{n+1} = ks\gamma_{n+1} + \beta_n,$$
$$\beta_{n+1} = ks\gamma_{n+1} + \alpha_n.$$

where $\alpha_0 = a, \beta_0 = b, \gamma_0 = c$ and $\gamma_1 = d$.

From these sequences, we generate the first few members of the sequences $\{\gamma_n\}^{\infty}_{n=0}$, $\{\alpha_n\}^{\infty}_{n=0}$ and $\{\beta_n\}^{\infty}_{n=0}$ with respect to n represented in Table 1, Table 2 and Table 3 respectively.

Table 1 This table shows first 8 members of $\{\gamma_n\}^{\infty}_{n=0}$ from the first set of sequences.

n	$\{\gamma_n\}^{\infty}_{n=0}$
0	c
1	d
2	$kd + c$
3	$k^2d + kc + d$
4	$k^3d + k^2c + c + 2kd + c$
5	$k^4d + k^3c + 3k^2d + 2kc + d$
6	$k^5d + k^4c + k^3d + 3k^2c + 3kd + c$
7	$k^6d + k^5c + 5k^4d + 4k^3c + 6k^2d + 3kc + d$

Table 2 This table shows first 8 members of $\{\alpha_n\}_{n=0}^{\infty}$ from the first set of sequences.

n	$\{\alpha_n\}_{n=0}^{\infty}$
0	a
1	$ksc + b$
2	$ks(c+d) + a$
3	$k^2sd + ks(2c+d) + b$
4	$k^3sd + k^2s(c+d) + ks(2c+2d) + a$
5	$k^4sd + k^3s(c+d) + k^2s(c+3d) + ks(3c+2d) + b$
6	$k^5sd + k^4s(c+d) + k^3s(c+4d) + k^2s(3c+3d) + ks(3c+3d) + a$
7	$k^6sd + k^5s(c+d) + k^4s(c+5d) + k^3s(4c+4d) + k^2s(3c+6d) + ks(4c+3d) + b$

Table 3 This table shows first 8 members of $\{\beta_n\}_{n=0}^{\infty}$ from the first set of sequences.

n	$\{\beta_n\}_{n=0}^{\infty}$
0	b
1	$ksc + a$
2	$ks(c+d) + b$
3	$k^2sd + ks(2c+d) + a$
4	$k^3sd + k^2s(c+d) + ks(2c+2d) + b$
5	$k^4sd + k^3s(c+d) + k^2s(c+3d) + ks(3c+2d) + a$
6	$k^5sd + k^4s(c+d) + k^3s(c+4d) + k^2s(3c+3d) + ks(3c+3d) + b$
7	$k^6sd + k^5s(c+d) + k^4s(c+5d) + k^3s(4c+4d) + k^2s(3c+6d) + ks(4c+3d) + a$

Theorem 2.1. For any positive integer k and n,

(a) $\gamma_n = F_{k,n}d + F_{k,n-1}c,$

(b) $\alpha_{2n} = (F_{k,2n} + F_{k,2n-1} -1)sd + (F_{k,2n-1} + F_{k,2n-2} + F_{k,2n} -1) sc + a,$

(c) $\beta_{2n} = (F_{k,2n} + F_{k,2n-1} -1)sd + (F_{k,2n-1} + F_{k,2n-2} + F_{k,2} -1)sc + b,$

(d) $\alpha_{2n-1} = (F_{k,2n-1} + F_{k,2n-2} -1)sd + (F_{k,2n-2} + F_{k,2n-3} + F_{k,2} -1)sc + b, for\ n \geq 2,$

(e) $\beta_{2n-1} = (F_{k,2n-1} + F_{k,2n-2} -1)sd + (F_{k,2n-2} + F_{k,2n-3} + F_{k,2} -1)sc + a, for\ n \geq 2.$

Proof. we will prove (a) by mathematical induction.

Let $P(n)$ be a statement $\gamma_n = F_{k,n}d + F_{k,n-1}c$ for $n \geq 1$, we will show that $P(1)$ is true.

Since $F_{k,1}d + F_{k,0}c = (1)d + (0)c = d = \gamma_1$, then $P(1)$ is true. Let $m \geq 1$, assume that $P(1), P(2),...,P(m-1), P(m)$ are true that is, $\gamma_n = F_{k,i}d + F_{k,i-1}c$, where $1 \leq i \leq m$.

We will show that $P(m+1)$ is true.

consider,

$$\begin{aligned}\gamma_{m+1} &= k\gamma_m + \gamma_{m-1}\\ &= k(F_{k,m}d + F_{k,m-1}c) + F_{k,m-1}d + F_{k,m-2}c\\ &= k(F_{k,m} + F_{k,m-1})d + (kF_{k,m-1} + F_{k,m-2})c\\ \gamma_{m+1} &= F_{k,m+1}d + F_{k,m}c.\end{aligned}$$

Then $P(m+1)$ is true.

By mathematical induction, the statement $P(n)$ is true for all $n \geq 1$.

Next, we will prove (b) by mathematical induction. Let $P(n)$ be a statement,

$$\alpha_{2n} = (F_{n-2n} + F_{k,2n-1} -1)sd + (F_{k,2n-1} + F_{k,2n-2} + F_{k,2} -1)sc + a, for\ n \geq 1.$$

We will show that $P(1)$ is true.

Now consider,

$$\begin{aligned}&(F_{k,2(n)} + F_{k,2(1)-1} -1)sd + (F_{k,2(1)-1} + F_{k,2(1)-2} + F_{k,2} -1)sc + a\\ &= (F_{k,2} + F_{k,1} -1)sd + (F_{k,1} + F_{k,0} + F_{k,2} -1)sc + a\\ &= (k+1-1)sd + (1+0+k-1)sc + a\\ &= ksd + ksc + a\\ &= ks(c+d) + a = \alpha_{2(1)}.\end{aligned}$$

Then $P(1)$ is true.

Let $m \geq 1$, assume that $P(m)$ is true that is,

$$\alpha_{2n} = (F_{k-2m} + F_{k,2n-1} -1)sd + (F_{k,2m-1} + F_{k,2m-2} + F_{k,2} -1)sc + a.$$

We will show that $P(m+1)$ is true.

Consider,

$$\begin{aligned}\alpha_{2m+2} &= ks\gamma_{2m+1} + \beta_{2m+1} \\ &= ks(F_{k,2m+1}d + F_{k,2m}c) + ks\gamma_{2m} + \alpha_{2m} \\ &= ks(F_{k,2m+1}d + F_{k,2m}c) + ks(F_{k,2m}d + F_{k,2m-1}c) \\ &\quad + (F_{k,2m} + F_{k,2m-1} - 1)sd + (F_{k,2m-1} + F_{k,2m-2} + F_{k,2} - 1)sc + a \\ &= [(kF_{k,2m+1} + F_{k,2m})sd + (kF_{k,2m} + F_{k,2m-1})sd - sd] + [(kF_{k,2m} + F_{k,2m-1})sc + (kF_{k,2m-1} + F_{k,2m-2})sc + F_{k,2}sc - sc] + a \\ &= (F_{k,2m+2} + F_{k,2m+1} - 1)sd + (F_{k,2m+1} + F_{k,2m} + F_{k,2} - 1)sc + a \\ \alpha_{2(m+1)} &= (F_{k,2(m+1)} + F_{k,2(m+1)-1} - 1)sd \\ &\quad + (F_{k,2(m+1)-1} + F_{k,2(m+1)-2} + F_{k,2} - 1)sc + a.\end{aligned}$$

Then $P(m+1)$ is true.

By mathematical induction the statement $P(n)$ is true for all $n > 1$.

The proof of (c) is similar to (b).

To prove equation (d) for $n \geq 2$, using (a) and (c) we have,

$$\begin{aligned}\alpha_{2n-1} &= ks\gamma_{2n-2} + \beta_{2n-2} \\ &= ks(F_{k,2n-2}d + F_{k,2n-3}c) + (F_{k,2n-2} + F_{k,2n-3} - 1)sd \\ &\quad + (F_{k,2n-3} + F_{k,2n-4} + F_{k,2} - 1)sc + b \\ &= [(kF_{k,2n-2} + F_{k,2n-3})sd + (F_{k,2n-2}sd - sd] + [(kF_{k,2n-3} + F_{k,2n-4})sc + (F_{k,2n-3}sc + F_{k,2}sc - sc] + b \\ &= (F_{k,2n-1} + F_{k,2n-2} - 1)sd + (F_{k,2n-2} + F_{k,2n-3} + F_{k,2} - 1)sc + b,\end{aligned}$$

then

$$\begin{aligned}\alpha_{2n-1} &= (F_{k,2n-1} + F_{k,2n-2} - 1)sd \\ &\quad + (F_{k,2n-2} + F_{k,2n-3} + F_{k,2} - 1)sc + b.\end{aligned}$$

is true.

By (a), (b), and the proof is similar to (d), then we have (e).

The proof is complete.

Next, we present the second sequences.

The second set of sequences has the form for $n \geq 0$,

$$\begin{aligned}\gamma_{n+2} &= k\gamma_{n+1} + \gamma_n, \\ \alpha_{n+1} &= k\gamma_{n+1} + \beta_n, \\ \beta_{n+1} &= k\gamma_{n+1} + \alpha_n.\end{aligned}$$

where $\alpha_0 = a$, $\beta_0 = b$, $\gamma_0 = c$ and $\gamma_1 = d$.

From these sequences, we generate the first 7 members of the sequences $\{\alpha_n\}^{\infty}_{n=0}$ and $\{\beta_n\}^{\infty}_{n=0}$ with respect to n represented in Table 4, and Table 5 respectively.

Table 4 This table shows first 7 members of $\{\alpha_n\}^{\infty}_{n=0}$ from the second set of sequences.

n	$\{\alpha_n\}^{\infty}_{n=0}$
0	a
1	$ksd + a$
2	$k^2sd + ks(c+d) + a$
3	$k^3sd + k^2s(c+d) + ks(c+2d) + b$
4	$k^4sd + k^3s(c+d) + k^2s(c+3d) + ks(2c+2d) + a$
5	$k^5sd + k^4s(c+d) + k^3s(c+4d) + k^2s(3c+3d) + ks(2c+3d) + b$
6	$k^6sd + k^5s(c+d) + k^4s(c+5d) + k^3s(4c+4d) + k^2s(3c+6d) + ks(3c+3d) + a$

Table 5 This table shows first 7 members of $\{\beta_n\}^{\infty}_{n=0}$ from the second set of sequences.

n	$\{\beta_n\}^{\infty}_{n=0}$
0	b
1	$ksd + a$
2	$k^2sd + ks(c+d) + b$
3	$k^3sd + k^2s(c+d) + ks(c+2d) + a$
4	$k^4sd + k^3s(c+d) + k^2s(c+3d) + ks(3c+2d) + b$
5	$k^5sd + k^4s(c+d) + k^3s(c+4d) + k^2s(3c+3d) + ks(2c+3d) + a$
6	$k^6sd + k^5s(c+d) + k^4s(c+5d) + k^3s(4c+4d) + k^2s(3c+6d) + ks(3c+3d) + b$

Theorem 2.2. For any positive integer k and n,

(a) $\gamma_n = F_{k,n}d + F_{k,n-1}c$,

(b) $\alpha_{2n} = (F_{k,2n+1} + F_{k,2n} -1)sd + (F_{k,2n} + F_{k,2n-1} -1)sc + a$,

(c) $\beta_{2n} = (F_{k,2n+1} + F_{k,2n} -1)sd + (F_{k,2n} + F_{k,2n-1} -1)sc + b$,

(d) $\alpha_{2n-1} = (F_{k,2n} + F_{k,2n-1} -1)sd + (F_{k,2n-1} + F_{k,2n-2} -1)sc + b$,

(e) $\beta_{2n-1} = (F_{k,2n} + F_{k,2n-1} -1)sd + (F_{k,2n-1} + F_{k,2n-2} -1)sc + a$.

Proof. The proofs are similar to theorem 2.1.

Finally, the last sequences in our work.

The third set of sequences has the form for $n \geq 0$,

$$\gamma_{n+1} = k\gamma_n + \frac{\alpha_n + \beta_n}{2s}$$
$$\alpha_{n+1} = ks\gamma_n + \beta_n,$$
$$\beta_{n+1} = ks\gamma_n + \alpha_n.$$

where $\alpha_0 = 2as$, $\beta_0 = 2sb$ and $\gamma_0 = c$.

The first 7 members of the sequences $\{\gamma_n\}^{\infty}_{n=0}$, $\{\alpha_n\}^{\infty}_{n=0}$ and $\{\beta_n\}^{\infty}_{n=0}$ are show in Table 6, Table 7, and Table 8 respectively.

Table 6 This table shows first 7 members of $\{\gamma_n\}^{\infty}_{n=0}$ from the third set of sequences.

n	$\{\gamma_n\}^{\infty}_{n=0}$
0	c
1	$kc + a + b$
2	$k^2c + k(a+b+c) + a + b$
3	$k^3c + k^2(a+b+2c) + k(2a+2b+c) + a + b$
4	$k^4c+ k^3(a+b+3c) + k^2(3a+3b+3c) + k(3c+3b+c) + a + b$
5	$k^5c + k^4(a+b+4c) + k^3(4c+4b+6c) + k^2(6a+6b+4c) + k(4a+4b+c) + a + b$
6	$k^6c + k^5(a+b+5c) + k^4(5a+5b+10c) + k^3(10a+10b+10c) + k^2(10a+10b+5c) + k(5a+5b+c) + a + b$

Table 7 This table shows first 7 members of $\{\alpha_n\}^{\infty}_{n=0}$ from the third set of sequences.

n	$\{\alpha_n\}^{\infty}_{n=0}$
0	$2as$
1	$ksc + 2bs$
2	$k^2sc + ks(a+b+c) + 2as$
3	$k^3sc + k^2s(a+b+2c) + ks(2a+2b+c) + 2bs$
4	$k^4sc+ k^3s(a+b+3c) + k^2s(3a+3b+3c) + ks(3a+3b+c) + 2as$
5	$k^5sc + k^4s(a+b+4c) + k^3s(4c+4b+6c) + k^2s(6a+6b+4c) + ks(4a+4b+c) + 2bs$
6	$k^6sc + k^5s(a+b+5c) + k^4s(5a+5b+10c) + k^3s(10a+10b+10c) + k^2s(10a+10b+5c) + ks(5a+5b+c) + 2as$

Table 8 This table shows first 7 members of $\{\beta_n\}^{\infty}_{n=0}$ from the third set of sequences.

n	$\{\beta_n\}^{\infty}_{n=0}$
0	$2bs$
1	$ksc + 2as$
2	$k^2sc + ks(a+b+c) + 2bs$
3	$k^3sc + k^2s(a+b+2c) + ks(2a+2b+c) + 2as$
4	$k^4sc+ k^3s(a+b+3c) + k^2s(3a+3b+3c) + ks(3a+3b+c) + 2bs$
5	$k^5sc + k^4s(a+b+4c) + k^3s(4c+4b+6c) + k^2s(6a+6b+4c) + ks(4a+4b+c) + 2as$
6	$k^6sc + k^5s(a+b+5c) + k^4s(5a+5b+10c) + k^3s(10a+10b+10c) + k^2s(10a+10b+5c) + ks(5a+5b+c) + 2bs$

Theorem 2.3. For any positive integer k and n,

(a) $\gamma_{n+1} = \gamma_n(F_{k,2}+F_{k,1}) = \gamma_1(F_{k,2}+F_{k,1})^n$,

(b) $\alpha_{2n} = \gamma_1 s(F_{k,2}+F_{k,1})^{2n-1} + as - bs$,

(c) $\beta_{2n} = \gamma_1 s(F_{k,2}+F_{k,1})^{2n-1} + bs - as$,

(d) $\alpha_{2n-1} = \gamma_1 s(F_{k,2}+F_{k,1})^{2n-2} + bs - as$,

(e) $\beta_{2n-1} = \gamma_1 s(F_{k,2}+F_{k,1})^{2n-2} + as - bs$.

Proof. To prove (a) we will show that $\gamma_{n+1} = \gamma_n(F_{k,2}+F_{k,1})$ since, $\gamma_{n+1} = k\gamma_n + \frac{\alpha_n+\beta_n}{2s}$ and we know that,

$$\frac{\alpha_n+\beta_n}{2s} = \frac{(ks\gamma_{n-1}+\beta_{n-1})+(ks\gamma_{n-1}+\alpha_{n-1})}{2s}$$

$$= k\gamma_{n-1} + \frac{\alpha_{n-1}+\beta_{n-1}}{2s},$$

so, we have $\gamma_{n+1} = k\gamma_n + k\gamma_{n-1} + \frac{\alpha_{n-1}+\beta_{n-1}}{2s}$

Since $\gamma_n = k\gamma_{n-1} + \frac{\alpha_{n-1}+\beta_{n-1}}{2s}$

we get that,

$$\gamma_{n+1} = k\gamma_n + \gamma_n$$

$$= \gamma_n (k+1)$$

$$\gamma_{n+1} = \gamma_n (F_{k,2}+F_{k,1}).$$

Next, we will show that $\gamma_{n+1} = \gamma_n (F_{k,2}+F_{k,1})^n$.

Since $\gamma_n = \gamma_{n-1} (F_{k,2}+F_{k,1})$

we have that,

$$\gamma_2 = \gamma_n (F_{k,2} + F_{k,1}).$$

$$\gamma_3 = \gamma_2 (F_{k,2} + F_{k,1}) = \gamma_1 (F_{k,2} + F_{k,1})^2.$$

$$\gamma_4 = \gamma_3 (F_{k,2} + F_{k,1}) = \gamma_1 (F_{k,2} + F_{k,1})^3.$$

.
.
.

$$\gamma_{n+1} = \gamma_1 (F_{k,2} + F_{k,1})^n.$$

thus $\gamma_{n+1} = \gamma_n (F_{k,2}+F_{k,1}) = \gamma_1 (F_{k,2} + F_{k,1})^n$.

We will prove (b) by mathematical induction.

Let $P(n)$ be the statement

$\alpha_{2n} = \gamma_1 s(F_{k,2}+F_{k,1})^{2n-1} + as - bs$ for $n \geq 1$.

We will show that $P(1)$ is true.

consider,

$$\gamma_1 s(F_{k,2}+F_{k,1})^{2(1)-1} + as - bs$$

$$= s(kc+a+b)(k+1) + as - bs$$

$$= k^2sc + ksa + ksb + ksc + as - bs + as + bs$$

$$= k^2sc + ks(a+b+c) + 2as + \alpha_{2(1)}$$

Then $P(1)$ is true.

Let $n \geq 1$, assume that $P(m)$ is true.

That is, $\alpha_{2m} = \gamma_1 s(F_{k,2}+F_{k,1})^{2m-1} + as - bs$.

We will show that $P(m+1)$ is true.

Consider,

$$\begin{aligned}
\alpha_{2(m+1)} &= \alpha_{2m+2} \\
&= ks\gamma_{2m+1} + \beta_{2m+1} \\
&= ks\gamma_{2m+1} + \beta_{2m} + \alpha_{2m} \\
&= ks\gamma_1(F_{k,2}+F_{k,1})^{2m} + ks\gamma_1(F_{k,2}+F_{k,1})^{2m-1} \\
&\quad + \gamma_1 s(F_{k,2}+F_{k,1})^{2m-1} + as - bs \\
&= ks\gamma_1(k+1)^{2m} + ks\gamma_1(k+1)^{2m-1} \\
&\quad + \gamma_1 s(k+1)^{2m-1} + as - bs \\
&= ks\gamma_1(k+1)(k+1)^{2m-1} + ks\gamma_1(k+1)^{2m-1} \\
&\quad + \gamma_1 s(k+1)^{2m-1} + as - bs \\
&= \gamma_1 s(k+1)^{2m-1} + [k(k+1)+k+1] + as - bs \\
&= \gamma_1 s(k+1)^{2m+1} + as - bs \\
&= \gamma_1 s(F_{k,2}+F_{k,1})^{2(m+1)-1} + as - bs
\end{aligned}$$

then $P(m+1)$ is true.

By mathematical induction the statement $P(n)$ is true for all $n \geq 1$.

The proof of (c) is similar to (b).

From (a) and (c) we have (d), and similarly from (a) and (b) we also have (e).

Conclusion and Discussion

A new three combined sequences related to *k*-Fibonacci sequences from new types were introduced and explicit formulas for their members are given.

From our sequences,

the first set of sequences,

$$\gamma_{n+2} = k\gamma_{n+1} + \gamma_n$$

$$\alpha_{n+1} = ks\gamma_n + \beta_n$$

$$\beta_{n+1} = ks\gamma_n + \alpha_n$$

the second set of sequences,

$$\gamma_{n+2} = k\gamma_{n+1} + \gamma_n$$

$$\alpha_{n+1} = ks\gamma_{n+1} + \beta_n$$

$$\beta_{n+1} = ks\gamma_{n+1} + \alpha_n$$

the third set of sequences,

$$\gamma_{n+1} = k\gamma_n + \frac{\alpha_n + \beta_n}{2s}$$
$$\alpha_{n+1} = ks\gamma_n + \beta_n,$$
$$\beta_{n+1} = ks\gamma_n + \alpha_n.$$

If $s = 1$, then the results correspond to the 3 set of sequences and the theorem 1.1, 1.2, and 1.3 in (Pakapongpun & Kongson, 2022). Other new schemes, modifying the standard form of *k*-Fibonacci sequences and new combined sequences will be discussed in the future.

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