ความสัมพันธ์ของกรีนบนโมนอยด์ของโคไฮเพอร์ซับสติติวชันเชิงเส้นชนิด $\tau = (n)$

Green's Relations on the Monoid of Linear Cohypersubstitutions of Type $\tau = (n)$

จุฑาลักษณ์ บุญศล[1], กิตติศักดิ์ แสงสุระ[2]

Julaluk Boonsol[1], Kittisak Saengsura[2]

Received: 24 August 2019 ; Revised: 1 October 2019 ; Accepted: 4 November 2019

บทคัดย่อ

โคไฮเพอร์ซับสติติวชันเชิงเส้นชนิด $\tau = (n)$ เป็นการส่งสัญลักษณ์การดำเนินการร่วมแบบ *n-ary* ไปยังพจน์ร่วมเชิงเส้นชนิด τ. สำหรับทุกโคไฮเพอร์ซับสติติวชันเชิงเส้น σ ชนิด $\tau = (n)$ ทำให้เกิดการส่ง $\hat{S}$ บนเซตของพจน์ร่วมเชิงเส้นชนิด τ. ทั้งหมด เซตของโคไฮเพอร์ซับสติติวชันเชิงเส้นชนิด τ ทั้งหมด ภายใต้การดำเนินการทวิภาค o_{coh} ซึ่งถูกกำหนดนิยามโดย $\sigma_1 o_{coh} \sigma_2 := \hat{\sigma}_1 o \sigma_2$ สำหรับทุก $\sigma_1, \sigma_2 \in Cohyp^{lin}(n)$ เป็นโมนอยด์ ในนี้เราจำแนกลักษณะความสัมพันธ์ของกรีนบน $Cohyp^{lin}(n)$.

คำสำคัญ: โคไฮเพอร์ซับสติติวชันเชิงเส้น พจน์ร่วมเชิงเส้น การซ้อนทับ ความสัมพันธ์ของกรีน

Abstract

Linear cohypersubstitutions of type $\tau = (n)$ are mappings which map the *n-ary* co-operation symbols to linear coterms of type τ. Every linear cohypersubstitution σ of type $\tau = (n)$ induces a mapping $\hat{S}$ on the set of all linear coterms of type τ. The set of all linear cohypersubstitutions of type τ under the binary operation o_{coh} which is defined by $\sigma_1 o_{coh} \sigma_2 := \hat{\sigma}_1 o \sigma_2$ for all $\sigma_1, \sigma_2 \in Cohyp^{lin}(n)$ forms a monoid. In this paper, we characterize Green's relations on $Cohyp^{lin}(n)$.

Keywords: linear cohypersubstitutions, linear coterms, superposition, Green's relations.

[1] นิสิตปริญญาโท, คณะวิทยาศาสตร์ มหาวิทยาลัยมหาสารคาม อำเภอกันทรวิชัย จังหวัดมหาสารคาม 44150

[2] ผู้ช่วยศาสตราจารย์, คณะวิทยาศาสตร์ มหาวิทยาลัยมหาสารคาม อำเภอกันทรวิชัย จังหวัดมหาสารคาม 44150

[1] Master degree student, Faculty of Science, Mahasarakham University, Kantharawichai District, Maha Sarakham 44150, Thailand.

[2] Asst. Prof., Faculty of Science, Mahasarakham University, Kantharawichai District, Maha Sarakham 44150, Thailand.

* Corresponding author ; Kittisak Saengsura, Faculty of Science, Mahasarakham University, Kantharawichai District, Maha Sarakham 44150, Thailand. Kittisak.s@mau.ac.th.

Julaluk Boonsol is supported by the Science Achievement Scholarship of Thailand (SAST). julaluk.bs@gmail.com.

Introduction

Let A be a non-empty set and n be a positive integer. The n-th copower $A^{\cup n}$ is the Cartesian product $A^{\cup n} := n \times A$, where $\underline{n}:=1,\ldots,n$. An element (i,a) in the copower corresponds to the element a in the i-th copy of A, for $1 \le i \le n$. A co-operation on A is a mapping $f^A: A \longrightarrow A^{\cup n}$ for some $n \ge 1$; the natural number n is called the arity of the co-operation f^A. We also need to recall that any n-ary co-operation f^A on set A can be uniquely expressed as a pair (f_1^A, f_2^A) of mappings, $f_1^A: A \longrightarrow \underline{n}$ and $f_2^A: A \longrightarrow A$; the first mapping gives the labeling used by f^A in mapping elements to copies of A, and the second mapping tells us what element of A is mapped to.

We shall denote by $cO_A^{(n)} = \{f^A \mid A \longrightarrow A^{\cup n}\}$ the set of all n-ary co-operations defined on A, and by $cO_A := \cup_{n\ge 1} cO_A^{(n)}$ he set of all finitary co-operations defined on A. An indexed coalgebra is a pair $(A ; (f_i^A)_{i\in l})$, where f_i^A is $\underline{a}$ n_i-ary co-operation defined on A, and $\tau = (n_i)_{i\in l}$ for $n_i \ge 1$ is called the type of the coalgebra. Coalgebras were studied by Drbohlav[1]. In[2], the following superposition of co-operations was introduced. If $f^A \in cO_A^{(n)}$ and $g_0^A,\ldots,g_{n-1}^A \in cO_A^{(k)}$ then the k-ary co-operation $f^A[g_0^A,\ldots,g_{n-1}^A]: A \longrightarrow A^{\cup k}$ is defined by $a \longmapsto ((g^A_{f_1^A(a)})_1(f_2^A(a)),(g^A_{f_1^A(a)})_2(f_2^A(a)))$ for all $a \in A$. The co-operation $f^A[g_0^A,\ldots,g_{n-1}^A]$ is called the superposition of f^A and $g_0^A,\ldots,g_{n-1}^A$. It will also be denoted by $comp_k^n(f^A, g_0^A,\ldots,g_{n-1}^A)$.

The injection co-operations $i_i^{n,A} : A \longrightarrow A^{\cup n}$ are special co-operations which are defined for each $0 \le i \le n-1$ by $i_i^{n,A} : A \longrightarrow A^{\cup n}$ with $a \longmapsto (i,a)$ for all $a \in A$. Then we get a multi-based algebra $((cO_A^{(n)})_{n\ge 1},(comp_k^n)_{k,n\ge 1},(i_i^{n,A})_{0\le i\le n-1})$, called the clone of co-operations on A. In[2], it is mentioned that this algebra is a clone, i.e. it satisfies the three clone axioms. In[3], K. Denecke and K. Saengsura gave a full proof of this fact and introduced the following coterms of type $\tau = (n_i)_{i\in l}$ were introduced. Let $(f_i)_{i\in l}$ be an indexed set of co-operation symbols such that for each $i \in l$. We say that symbol f_i has arity n_i, for $i \in l$. Let $U\{e_i^n \mid n \ge 1, n \in N, 0 \le j \le n-1\}$ be a set of symbols which is disjoint from the set $\{fi \mid i \in l\}$. We assign to each e_j^n the positive integer n as its arity. Then coterms of type τ are defined as follows:

(i) For every $i \in l$, the co-operation symbol f_i is an n_i-ary coterm of type τ.

(i) For every $n \ge 1$ and $0 \le j \le n-1$, the symbol e_j^n is an n-ary coterm of type τ.

(i) If $t_1,\ldots,t_{n_i}$ are n-ary coterms of type τ, then $f_i[t_1,\ldots,t_{n_i}]$ is an n-ary coterm of type τ and if $t_0,\ldots,t_{n-1}$ are m-ary coterms of type τ, then $e_j^n[t_0,\ldots,t_{n-1}]$ is an m-ary coterm of type τ, for every $i \in l$ and $n \ge 1$ and $0 \le j \le n-1$.

Let $cT_t^{(n)}$ be the set of all n-ary coterms of type τ and let $cT_t := \bigcup_{n\ge 1} cT_t^{(n)}$ be the set of all (finitary) coterms of type τ.

Definition 1.1 Let $t \in cT_t$ be a coterm and $E(t) = \{e_i^n \mid e_i^n$ occurs in t and $0 \le i \le n-1$. Then t is a linear coterm if for each $e_i^n \in E(t)$, e_i^n occurs only once in t.

We denote by $cT_t^{lin,(n)}$ the set of all n-ary linear coterms of type τ and $cT_t^{lin} := \bigcup_{n\ge 1} cT_t^{lin,(n)}$ the set of all (finitary) linear coterms of type τ.

We define a family of superposition operations $(\bar{S}_m^n)_{m,n\ge 1}$ on this sequence, as follows.

Definition 1.2 The operation $\bar{S}_m^n : cT_t^{lin,(n)} \times (cT_t^{lin,(m)})^n \longrightarrow cT_t^{lin,(m)}$ is defined by induction on the complexity of linear coterm definition, as follows:

(i) If e_i^n is an n-ary linear coterm of type τ, $t_0,\ldots,t_{n-1}$ are m-ary linear coterms of type τ for $0 \le j \le n-1$ and $E(t_j) \cap E(t_k) = \emptyset$ for $j,k \in \{0,\ldots,n-1\}$ and $j \ne k$, then $\bar{S}_m^n(e_i^n,t_0,\ldots,t_{n-1}) := t_i$ is an m-ary linear coterm of type τ.

(ii) If f is an n-ary linear coterm of type τ, $t_1,\ldots,t_n$ are m-ary linear coterms of type τ and $E(t_j) \cap E(t_k) = \emptyset$ for $j,k \in \{1,\ldots,n\}$, then $\bar{S}_m^n(f,t_t,\ldots,t_n) := f[t_t,\ldots,t_n]$ is an n-ary linear coterm of type τ.

(iii) If f is an n-ary co-operation symbol, $S_1,\ldots,S_n$ are n-ary linear coterms of type τ where $E(s_j) \cap E(s_k) = \emptyset$ for $j,k \in \{1,\ldots,n\}$ and $t_1,\ldots,t_n$ are m-ary linear coterms of type τ where $E(t_j) \cap E(t_k) = \emptyset$ for $j,k \in \{1,\ldots,n\}$, then $\bar{S}_m^n(f[s_1,\ldots,s_n],t_t,\ldots,t_n) := f[\bar{S}_m^n(s_1,t_1,\ldots,t_n),\ldots,\bar{S}_m^n(s_n,t_1,\ldots,t_n)]$ is an n-ary linear coterm of type t.

Together with these operations we obtain a heterogeneous algebra $cT_t^{lin}:=((cT_t^{lin,(n)})_{n\geq 1}, (\bar{S}_m^{\,n})_{m,n\geq 1}, (e_j^n)_{0\leq j\leq n-1})$.

Definition 1.3 A linear cohypersubstitution of type t is a mapping $S : \{ f \}\rightarrow cT_t^{lin}$ from the set of all co-operation symbols to the set of all linear coterms which is inductively defined by the following steps:

(i) $\hat{\sigma}[e_j^n]:= e_j^n$ for every $n \geq 1$ and $0 \leq j \leq n-1$,

(ii) $\hat{\sigma}[f]:= \sigma[f]$,

(iii) $\hat{\sigma}[f[t_1,...,t_n]]:= \bar{S}_n^{\,n} (\sigma(f),\hat{\sigma}[t_1],...,\hat{\sigma}[t_n])$ and assume that $\hat{\sigma}[t_j]$ is already defined and $E(t_j)$ are distinct for all $1 \leq j \leq n$.

Let $Cohyp^{lin}(\tau)$ be the set of all linear cohypersubstitutions of type τ. Since the extension of a linear cohypersubstitution of type τ maps cT_τ^{lin} to cT_τ^{lin}, we may define a binary operation o_{coh} by $\hat{\sigma}_1\, o_{coh}\, \sigma_2:=\hat{\sigma}_1\, o\, \sigma_2$ where o is the usual composition of mappings. Let σ_{id} be the linear cohypersubstitution defined by $\sigma_{id}(f): = f$.

In 2016, D. Boonchari and K. Saengsura studied the monoid of cohypersubstitutions of type $\tau=(n)^4$. In this paper, we characterize Green's relations on $Cohyp^{lin}(n)$.

Main results

In this section, we obtain the linear cohypersubstitutions σ_t and σ_s which are R-related, L-related, H-related, D-related and J-related as following theorem:

We characterize the Green's relation R on $Cohyp^{lin}(n)$ and we recall the definition of Green's relation R i.e., let a, b be elements of semigroup S. Then $a\, R\, b$ if and only if there exists x,y in S such that $xa=b, yb=a$.

Theorem 2.1 Let $\sigma_t, \sigma_s\in Cohyp^{lin}(n)$. If $t = e_i^n$, $s = e_j^n\in cT_t^{lin,(n)}$ for all $i,j\in\{0,...,n-1\}$ then $\sigma_t\, R\, \sigma_s$.

Proof Assume that $t = e_i^n$, $s = e_j^n\in cT_t^{lin,(n)}$ for all $i,j\in\{0,...,n-1\}$. We will show that there are σ_r, $\sigma_w\in Cohyp^{lin}(n)$ such that $\sigma_t = \sigma_s\, o_{coh}\, \sigma_r$ and $\sigma_s = \sigma_t\, o_{coh}\, \sigma_w$.

Since $\sigma_s(f) = s = e_j^n$ and $\hat{\sigma}_t[e_j^n]=e_j^n$, then $\sigma_s(f) = e_j^n$

$= \hat{\sigma}_t[e_j^n]$

$= \hat{\sigma}_t[\sigma_{e_j^n}(f)]$

$= \hat{\sigma}_t[\sigma_s(f)]$

$= (\sigma_t\, o_{coh}\, \sigma_s)(f)$.

Therefore, $\sigma_s = \sigma_t\, o_{coh}\, \sigma_s$.

Similarly, one can show that $\sigma_t = \sigma_s\, o_{coh}\, \sigma_r$ for some $\sigma_r\in Cohyp^{lin}(n)$.

This implies that $\sigma_t\, R\, \sigma_s$.

Theorem 2.2 Let $\sigma_t,\sigma_s\in Cohyp^{lin}(n)$. If $t = f[e^n_{jn_i}, ..., e^n_{jn_i}]\in cT_t^{lin,(n)}$ and $s = f[e^n_{j_0},..., e^n_{j_{n-1}}]\in cT_t^{lin,(n)}$ where $i_o,...,i_{n-1}, j_o,...,j_{n-1}\in\{0,...,j_{n-1}\}$ then $\sigma_t\, R\, \sigma_s$.

Proof Let $r = f[r_1,...,r_n]\in cT_t^{lin,(n)}$ such that $r_{j_k} = e^n_{i_k}$ for all $j_k\in\{0,...,n-1\}$ and $k = 0,...,n-1$.

Then $\sigma_t(f) = f[e^n_{j_0},..., e^n_{j_{n-1}}]$, and so $(\sigma_s\, o_{coh}\, \sigma_r)(f) = [\hat{\sigma}_r(f)]$

$= \hat{\sigma}_s[f\,[r_1,...,r_n]]$

$= \sigma_s(f)[r_1,...,r_n]$

$= (f[e^n_{j_0},..., e^n_{j_{n-1}}][r_1,...,r_n]$

$= f[e^n_{j_0}[r_1,...,r_n],..., e^n_{j_{n-1}}[r_1,...,r_n]]$

$= f[e^n_{i_0},..., e^n_{i_{n-1}}]$

$= t$

$= \sigma_t(f)$.

Therefore, $\sigma_s\, o_{coh}\, \sigma_r = \sigma_t$.

Similarly, one can show that $\sigma_s = \sigma_t\, o_{coh}\, \sigma_w$ for some $\sigma_w\in Cohyp^{lin}(n)$.

Hence, $\sigma_t\, R\, \sigma_s$.

Therefore, $(\sigma_t, \sigma_s)\in R$.

For linear cohypersubstitutions σ_t, σ_s such that t and s are different form i.e., $t\in\{e_i^n \mid 0 \leq i \leq n-1\}$ and $s\in cT_t^{lin,(n)}\backslash\{e_i^n \mid 0 \leq i \leq n-1\}$, we have that $(\sigma_t, \sigma_s)\notin R$ as the following example:

Example 2.3 Let $\sigma_t, \sigma_s\in Cohyp^{lin}(n)$ and $t = e^n_i$, $s = f[e^n_{j_0},...,e^n_{j_{n-1}}]\in cT_t^{lin,(n)}$ for all $i, J_o,..., J_{n-1}\in\{0,..., n-1\}$ and $E(s)$ be distinct.

Assume that $(\sigma_t, \sigma_s)\in R$.

Then there is $\sigma_w\in Cohyp^{lin}(n)$. such that $\sigma_s = \sigma_t\, o_{coh}\, \sigma_w$.

Hence

$f[e^n_{j_0},...,e^n_{j_{n-1}}] = s$

$= \sigma_s(f)$

$= \hat{\sigma}_t[\sigma_w(f)]$

$= \hat{\sigma}_t[w]$.

But we cannot find $w \in cT_t^{lin,(n)}$ such that

$\hat{\sigma}_t[w] = f[e^n_{j_1},...,e^n_{j_n}]$.

So $(\sigma_t, \sigma_s) \in R$.

Remark The number of pairs (σ_t, σ_s) in which $\sigma_t R \sigma_s$ is $n^2 + (n!)^2$.

Next, we characterize the Green's relation L on $Cohyp^{lin}(n)$ and we recall the definition of Green's relation L i.e., $a L b$ if and only if there exists u, v in S such that $au = v, bv = u$.

Theorem 2.4 Let $\sigma_t, \sigma_s \in Cohyp^{lin}(n)$ and $t, s \in \{e_i^n \mid n \geq 1, 0 \leq i \leq n\text{-}1\}$. If $\sigma_t L \sigma_s$, then $t = s$.

Proof Assume that $\sigma_t L \sigma_s$.

Then there are $\sigma_u, \sigma_v \in Cohyp^{lin}(n)$ such that $\sigma_t = \sigma_u \, o_{coh} \, \sigma_s$ and $\sigma_s = \sigma_v \, o_{coh} \, \sigma_t$.

Let $\sigma_t(f) = t = e_j^n$ and $\sigma_s(f) = s = e_j^n$.

Then

$e_i^n = t$

$= \sigma_t(f)$

$= \hat{\sigma}_u[\sigma_s(f)]$

$= \hat{\sigma}_u[e_i^n]$

$= e_i^n$

$= s$.

Therefore, $t = s$.

For linear cohypersubstitutions σ_t, σ_s such that $t, s \in \{e_i^n \mid 0 \leq i \leq n\text{-}1\}$. and $t \neq s$, we have that $(\sigma_t, \sigma_s) \in L$. as the following example:

Example 2.5 Let $\sigma_t, \sigma_s \in Cohyp^{lin}(n)$

Assume that $t = e_i^n$, $s = e_j^n \in cT_t^{lin,(n)}$ for all $i,j \in \{0,...,n\text{-}1\}$ and $i \neq j$.

Then $e_i^n = t = \sigma_t(f)$ and $e_j^n = s = \sigma_s(f)$.

Since for all $\sigma_s \in Cohyp^{lin}(n)$, we have that $\hat{\sigma}_u[e_j^n] = e_j^n$. Then $\hat{\sigma}_u[\sigma_s(f)] = \sigma_s(f) \neq \sigma_t(f)$.

Therefore, $(\sigma_t, \sigma_s) \in L$.

Theorem 2.6 If $t = f[e^n_{i_o}, ..., e^n_{i_{n-1}}] \in cT_t^{lin,(n)}$ and $s = f[e^n_{j_o}, ..., e^n_{j_{n-1}}] \in cT_t^{lin,(n)}$ where $i_o,...,i_{n-1}, j_o,...,j_{n-1} \in \{0,...,n\text{-}1\}$, then $\sigma_t L \sigma_s$.

Proof Let $v = f[v_1,...,v_n] \in cT_t^{lin,(n)}$ such that $v_1,...,v_n \in \{e_i^n \mid i = 0,...,n\text{-}1\}$ and $v_1[e^n_{i_0},...,e^n_{i_{n-1}}] = e^n_{j_0},...,e_n[e^n_{i_o},...,e^n_{i_{n-1}}] = e^n_{j_{n-1}}$.

Then

$\hat{\sigma}_v[\sigma_t(f)] = \hat{\sigma}_v[f[e^n_{i_0},...,e^n_{i_{n-1}}]]$

$= \sigma_v(f)[e^n_{i_0},...,e^n_{i_{n-1}}]$

$= (f[v_1,...,v_n])[e^n_{i_0},...,e^n_{i_{n-1}}]$

$= f[v_1[e^n_{i_0},...,e^n_{i_{n-1}}] = e^n_{j_0},...,v_n[e^n_{i_0},...,e^n_{i_{n-1}}]]$

$= f[e^n_{j_0},...,e^n_{j_{n-1}}]$

$= s$

$= \sigma_s(f)$.

Therefore, $\sigma_v \, o_{coh} \, \sigma_t = \sigma_s$.

Similarly, one can show that $\sigma_t = \sigma_u \, o_{coh} \, \sigma_s$ for some $\sigma_u \in Cohyp^{lin}(n)$.

Hence, $\sigma_t L \sigma_s$.

Remark The number of pairs (σ_t, σ_s) in which $\sigma_t L \sigma_s$ is $n + (n!)^2$.

Next, we characterize the Green's relation H on $Cohyp^{lin}(n)$.

Theorem 2.7 Let $\sigma_t, \sigma_s \in Cohyp^{lin}(n)$ and $t, s \in \{e_i^n \mid n \geq 1, 0 \leq i \leq n\text{-}1\}$. Then $\sigma_t H \sigma_s$ if and only if $t = s$.

Proof Assume that $\sigma_t H \sigma_s$.

Then $\sigma_t H \sigma_s$ and $\sigma_t R \sigma_s$.

By Theorem 2.4, we get that $t = s$.

Similarly, assume that $t = s$.

Then $\sigma_t = \sigma_s$.

Since and are equivalence relations,

we have $\sigma_t L \sigma_s$ and $\sigma_t R \sigma_s$.

Therefore, $\sigma_t H \sigma_s$.

Theorem 2.8 Let $t, s \in cT_t^{lin,(n)} \setminus \{e_i^n \mid n \geq 1, 0 \leq i \leq n\text{-}1\}$. Then $\sigma_t H \sigma_s$.

Proof Let $t = f[e^n_{i_o}, ..., e^n_{i_{n-1}}] \in cT_t^{lin,(n)}$ and $s = f[e^n_{j_o}, ..., e^n_{j_{n-1}}] \in cT_t^{lin,(n)}$ for $i_o,...,i_{n-1}, j_o,...,j_{n-1} \in \{0,...,n\text{-}1\}$.

By Theorem 2.2, we have that $\sigma_t R \sigma_s$.

By Theorem 2.6, we have that $\sigma_t L \sigma_s$.

Therefore, $\sigma_t H \sigma_s$.

Remark The number of pairs (σ_t, σ_s) in which $\sigma_t H \sigma_s$ is $n + (n!)^2$.

Next, we characterize the Green's relation D on $Cohyp^{lin}(n)$.

Theorem 2.9 Let $(\sigma_t, \sigma_s) \in Cohyp^{lin}(n)$ and $t, s \in \{e_i^n \mid n \geq 1, 0 \leq i \leq n\text{-}1\}$. Then $\sigma_t D \sigma_s$.

Proof Since $\sigma_t L \sigma_t$ and by Theorem 2.2,

we have that $\sigma_t R \sigma_t$.

Then $\sigma_t D \sigma_t$.

Theorem 2.10 Let $t, s \in cT_t^{lin,(n)} \setminus \{e_i^n \mid n \geq 1, 0 \leq i \leq n\text{-}1\}$. Then $\sigma_t D \sigma_s$.

Proof Let $t = f[e^n_{i_o}, ..., e^n_{i_{n-1}}] \in cT_t^{lin,(n)}$ and $s = f[e^n_{j_o}, ..., e^n_{j_{n-1}}] \in cT_t^{lin,(n)}$ for $i_o,...,i_{n-1}, j_o,...,j_{n-1} \in \{0,...,n\text{-}1\}$.

By Theorem 2.2, we have that $\sigma_t R \sigma_s$.

By Theorem 2.6, we get that $\sigma_t L \sigma_s$.

Therefore, $\sigma_t D \sigma_s$.

For linear cohypersubstitutions σ_t, σ_s such that t and s are different form i.e., $t \in \{e_i^n \mid 0 \leq i \leq n\text{-}1\}$ and $s \in cT_t^{lin,(n)} \setminus \{e_i^n \mid 0 \leq i \leq n\text{-}1\}$, we have that $(\sigma_t, \sigma_s) \notin D$ as the following example:

Example 2.11 Let $\sigma_t, \sigma_s \in Cohyp^{lin}(n)$ and $t = e_i^n$, $s = f[e^n_{j_o}, ..., e^n_{j_{n-1}}] \in cT_t^{lin,(n)}$ for all $i, j_o,...,j_{n-1} \in \{0,...,n\text{-}1\}$ and $E(s)$ be distinct.

Then $\sigma_t(f) = e_i^n$ and $\sigma_s(f) = f[e^n_{j_0},...,e^n_{j_{n-1}}]$.

By Theorem 2.4, we get that $\sigma_t L \sigma_s$.

But by Theorem 2.3, we have that $(\sigma_t, \sigma_s) \notin R$.

Hence, $(\sigma_t, \sigma_s) \notin D$.

Remark The number of pairs (σ_t, σ_s) in which $\sigma_t D \sigma_s$ is $n^2 + (n!)^2$.

Next, we characterize the Green's relation J on $Cohyp^{lin}(n)$.

Theorem 2.12 Let $(\sigma_t, \sigma_s) \in Cohyp^{lin}(n)$ and $t, s \in \{e_i^n \mid n \geq 1, 0 \leq i \leq n\text{-}1\}$. Then $\sigma_t J \sigma_s$.

Proof Let $t = e_i^n$, $s = e_j^n$ and $u \in cT_t^{lin,(n)}$.

Since $\hat{\sigma}_u[e_k^n] = e_k^n$ for all $k = 0,...,n\text{-}1$.

we have

$\sigma_t(f) = e_i^n$

$= \hat{\sigma}_s[e_i^n]$

$= \hat{\sigma}_u[\hat{\sigma}_s[e_i^n]]$

$= \hat{\sigma}_u[\hat{\sigma}_s[\sigma_{e_i^n}(f)]]$

$= \hat{\sigma}_u[\hat{\sigma}_s[\sigma_t(f)]]$.

Therefore, $\sigma_t = \sigma_u \circ_{coh} \sigma_s \circ_{coh} \sigma_t$.

Similarly, one can show that $\sigma_s = \sigma_x \circ_{coh} \sigma_t \circ_{coh} \sigma_y$ for some $\sigma_x, \sigma_y \in Cohyp^{lin}(n)$.

Hence, $\sigma_t J \sigma_s$.

Theorem 2.13 Let $t, s \in cT_t^{lin,(n)} \setminus \{e_i^n \mid n \geq 1, 0 \leq i \leq n\text{-}1\}$. Then $\sigma_t J \sigma_s$.

Proof Let $t = f[e^n_{i_o}, ..., e^n_{i_{n-1}}] \in cT_t^{lin,(n)}$ and $s = f[e^n_{j_o}, ..., e^n_{j_{n-1}}] \in cT_t^{lin,(n)}$ for $i_o,...,i_{n-1}, j_o,...,j_{n-1} \in \{0,...,n\text{-}1\}$.

We let $r = f[r_1,...,r_n]$ such that $r_{j_k} = e^n_{j_k}$ where $j_k \in \{0,...,n\text{-}1\}$ and $k = 0,...,n\text{-}1$.

By Theorem 2.2, we get that $\sigma_t(f)] = \hat{\sigma}_s[\sigma_r(f)]$.

Let $v = (f)[e^n_o,...,e^n_{n-1}] \in cT_t^{lin,(n)}$.

Then

$\hat{\sigma}_v[\sigma_t(f)] = \hat{\sigma}_v[f[e^n_{i_0},...,e^n_{i_{n-1}}]]$

$= \sigma_v(f)[e^n_{i_0},...,e^n_{i_{n-1}}]$

$= (f[v_1,...,v_n])[e^n_{i_0},...,e^n_{i_{n-1}}]$

$= f[v_1[e^n_{i_0},...,e^n_{i_{n-1}}] = e^n_{j_0},...,v_n[e^n_{i_0},...,e^n_{i_{n-1}}]]$

$= f[e^n_{j_0},...,e^n_{j_{n-1}}]$

$= t$

$= \sigma_t(f)$.

Therefore, $\sigma_v \circ_{coh} \sigma_s \circ_{coh} \sigma_r = \sigma_t$.

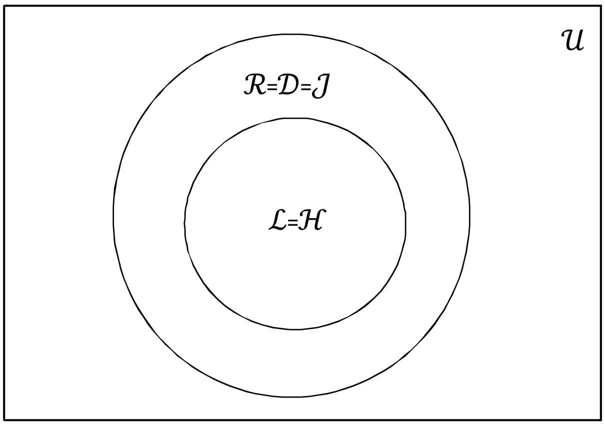
Similarly, one can show that $\sigma_s = \sigma_x \circ_{coh} \sigma_t \circ_{coh} \sigma_y$ for some $\sigma_x, \sigma_y \in Cohyp^{lin}(n)$.

Hence, $\sigma_t J \sigma_s$.

Remark The number of pairs (σ_t, σ_s) in which $\sigma_t J \sigma_s$ is $n^2 + (n!)^2$.

We conclude the R, L, H, D and J as the following diagram:

$U = \{(\sigma_t, \sigma_s) \mid t, s \in cT_t^{lin,(n)}\}$.



If $t, s \in \{e_i^n \mid n \ge 1, 0 \le i \le n\text{-}1\}$ and $t = s$ in L, then $L \subseteq R$.

Acknowledgements

The authors would like to thank the referee for useful remarks. We also would like to thank the Faculty of Science of Mahasarakham University Thailand and the Science Achievement Scholarship of Thailand (SAST) for the financial support.

References

1. Drbohlav K. On quasicovarieties. Acta Fac. Rerum Natur. Univ. Comenian. Math. Mimoriadne Cislo. 1971 ; 17-20.
2. Csa'ka'ny B. Completeness in coalgebras. Acta Sci. Math. 1985 ; 48: 75-84.
3. Denecke K, Saengsura K. Menger Algebras and Clones of Cooperations. Algebra Colloquium. 2008 ; 15(2): 223-234.
4. Boonchari D, Saengsura K. Monoid of Cohypersubstitutions of type t = (*n*). Thai Journal of Mathematics. 2016 ; 14: 191-201.
5. Denecke K. The partial clone of linear terms. Siberian Mathematical Journal. 2016 ; 57(4): 589-598
6. Denecke K, Lau D, Poschel R, Schweigert D. Hyperidentities, hyperequational classes and clone congruences. Contribution to General Algebra. 2002 ; 7: 97-118.
7. Denecke K, Saengsura K. Cohyperidentities and M-solid classes of coalgebras. Discrete Mathematics. 2009 ; 304(4): 772-783.
8. Denecke K, Wismath SL. Universal algebra and applications in theoretical computer science. Boca Raton, Chapman&Hall/CRC. 2002.
9. Howie JM. Fundamentals of Semigroup Theory. Oxford Science Publications, Clarendon Press. Oxford. 1995.
10. Jermjitporn S, Saengsura N. Generalized cohypersubstitutions of type $\cdot = (\cdot_i)_{i \in i}$ Thai Journal of Mathematics. 2013 ; 4: 747-755.
11. Koppitz J, Denecke K. M-solid varieties of algebras. Springer Science+Business Media. Inc. 2006.
12. Leeratanavalee S, Denecke K. Generalized hypersubstitutions and strongly solid varieties. Proceedings of the "59th Workshop on General Algebra", Conference for Young Algebraists. Potsdam. 200 ; 135-145.