อัตราส่วนของลำดับย่อยของจำนวนฟีโบนักชีที่มีดัชนีเป็นเลขชี้กำลัง n

The ratio of the n-th exponential subsequence of the Fibonacci Sequence

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บทคัดย่อ

เป็นที่ทราบกันดีว่าอัตราส่วนของพจน์ที่ติดกันของจำนวนฟีโบนักชี $\{F_m\}_{m=0}^{\infty}$ และอัตราส่วนของพจน์ที่ติดกันของจำนวนลูคัส $\{L_m\}_{m=0}^{\infty}$ ลู่เข้าสู่อัตราส่วนทองคำ งานวิจัยนี้ศึกษาลำดับย่อย $\{F_{m^n}\}_{m=0}^{\infty}$ เมื่อ n เป็นจำนวนเต็มบวก โดยได้แสดงว่าลิมิตของอัตราส่วนระหว่าง $\frac{F_{(m+1)^n}}{F_{m^n}}$ และ $\frac{F_{m^n}}{F_{(m-1)^n}}$ ลู่เข้าก็ต่อเมื่อ $n \le 2$ โดยทำการพิสูจน์ลำดับที่เกิดจากความสัมพันธ์เวียนเกิดอันดับสองในรูปทั่วไปที่ครอบคลุมลำดับฟีโบนักชี นอกจากนี้ยังได้ให้ค่าของลิมิตที่เกิดขึ้น

คำสำคัญ: ลำดับฟีโบนักชี อัตราส่วน การลู่เข้า ความสัมพันธ์เวียนเกิด ลำดับย่อยที่มีดัชนีเป็นเลขชี้กำลังเป็น n

Abstract

It is well known that the ratios of the consecutive terms of the Fibonacci numbers $\{F_m\}_{m=0}^{\infty}$ and those of the Lucas numbers $\{L_m\}_{m=0}^{\infty}$ converge to the golden ratio. In this work, we study the n-exponential subsequence $\{F_{m^n}\}$, where n is a positive integer. We show that the limit of the quotient between $\frac{F_{(m+1)^n}}{F_{m^n}}$ and $\frac{F_{m^n}}{F_{(m-1)^n}}$ converges if and only if $n \le 2$ by proving a more general statement for the sequences satisfying a recurrence relation of order 2 that covers the Fibonacci sequence. We also give the limit of the convergence if it exists.

Keyword: Fibonacci sequence, Quotient, Convergence, Recurrence relation, n-exponential subsequence

Introduction

The *Fibonacci sequence* $\{F_m\}_{m=0}^{\infty}$ is defined by the recurrence relation

$$F_m = F_{m-1} + F_{m-2}, \text{ for } m \ge 2, \qquad (1)$$

where $F_0 = 0$ and $F_1 = 1$. In 2015, Craciun[1] defined a geometrical generalization of the golden ratio by considering a ratio between two sub-segments and its relation to a homogeneous function M defined by

$$M:(0,\infty)\times(0,\infty)\to(0,\infty)$$

satisfying

i. $x < M(x,y) < y$, for all $0 < x < y$

and

ii. $M(\lambda x, \lambda y) = \lambda M(x,y)$, for all $\lambda, x, y \in (0,\infty)$.

The Fibonacci numbers have been generalized in many ways, one of which is the $k-$Fibonacci numbers[2] defined by, for a non-zero integer k,

$$F_{k,m} = kF_{k,m-1} + F_{k,m-2}, \text{ for } m \ge 2,$$

where $F_{k,0} = 0$ and $F_{k,1} = 1$. It is well known that the ratio of consecutive Fibonacci numbers converges to the golden ratio $\varphi = \frac{1+\sqrt{5}}{2}$. If we consider the n-exponential subsequence $\{F_{m^n}\}_{m=1}^{\infty}$ of the Fibonacci sequence, it is obvious that the ratio of consecutive terms goes to infinity.

We will study a more generalized form of the Fibonacci and $k-$Fibonacci numbers. For a non-zero

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real number k, we let $\{a_m\}_{m=0}^{\infty}$ be a sequence generated by a recurrence relation

$$a_m = ka_{m-1} + a_{m-2}, \text{ for } m \geq 2 \quad (2)$$

where $a_0 = s$ and $a_1 = t$. We assume that $a_{m^n} \neq 0$, for all $m, n \geq 1$.

The followings are examples of the sequences satisfying (2):

- if $k=1, s=0, t=1,$ then a_n is the Fibonacci number F_n,
- if $k=1, s=2, t=1,$ then a_n is the Lucas number L_n,
- if $k=2, s=0, t=1,$ then a_n is the Pell number P_n,
- if $k=2, s=2, t=2,$ then a_n is the Pell-Lucas number Q_n.

In this paper, we are interested in the growth rate of such ratios which is the quotient of $\frac{a_{(m+1)^n}}{a_{m^n}}$ and $\frac{a_{m^n}}{a_{(m-1)^n}}$. It has been shown that if k is a positive integer, then[3,4]

$$\lim_{m\to\infty} \frac{F_{k,m+p}}{F_{k,m}} = \varphi_k^p, \quad (3)$$

where p is a positive integer and $\varphi_k = \frac{k+\sqrt{k^2+4}}{2}$. By (3), it can be verified that the limit of the quotient of $\frac{F_{k,(m+1)^n}}{F_{k,m^n}}$ and $\frac{F_{k,m^n}}{F_{k,(m-1)^n}}$ converges if and only if $n \leq 2$, and that if $n = 2$, then the limit converges to φ_k^2. Considering a more generalized sequence $\{a_m\}_{m=0}^{\infty}$ we give a result related to the quotient of $\frac{a_{(m+1)^n}}{a_{m^n}}$ and $\frac{a_{m^n}}{a_{(m-1)^n}}$ in

Theorem 2.1.

In 2016, R. Euler and J. Sadek[5] showed that

$$a_m = \frac{1}{r_1 - r_2}\left(\alpha r_1^m - \beta r_2^m\right), \quad (4)$$

where $\alpha = s - tr_2$, $\beta = t - sr_1$ and

$$r_1, r_2 \in \left\{\frac{k+\sqrt{k^2+4}}{2}, \frac{k-\sqrt{k^2+4}}{2}\right\}$$ such that $|r_1| > |r_2|$. We note that $0 < |r_2| < 1$.

Main Theorem

Considering $\{a_n\}_{n=0}^{\infty}$ satisfying (2),

we let

$$a_m^{(n)} = \frac{a_{m^n}}{a_{(m-1)^n}}.$$

Theorem 2.1.

$$\lim_{m\to\infty} \frac{a_{m+1}^{(n)}}{a_m^{(n)}} = \begin{cases} 0, & \text{if } n = 1, \\ r_1^2, & \text{if } n = 2, \\ \infty, & \text{otherwise.} \end{cases}$$

Proof. By using the Binet formula of a_m in (4),

$$\frac{a_{m+1}^{(n)}}{a_m^{(n)}}$$

$$= \frac{a_{(m+1)^n}}{a_{m^n}} \cdot \frac{a_{(m-1)^n}}{a_{m^n}}$$

$$= \frac{\alpha r_1^{(m+1)^n} - \beta r_2^{(m+1)^n}}{\alpha r_1^{m^n} - \beta r_2^{m^n}} \cdot \frac{\alpha r_1^{(m-1)^n} - \beta r_2^{(m-1)^n}}{\alpha r_1^{m^n} - \beta r_2^{m^n}}$$

$$= \frac{\alpha^2 r_1^{(m+1)^n+(m-1)^n} - \alpha\beta r_2^{(m-1)^n} r_1^{(m+1)^n}}{\left(\alpha r_1^{m^n} - \beta r_2^{m^n}\right)^2}$$

$$+ \frac{-\alpha\beta r_1^{(m-1)^n} r_2^{(m+1)^n} + \beta^2 r_2^{(m+1)^n+(m-1)^n}}{\left(\alpha r_1^{m^n} - \beta r_2^{m^n}\right)^2}$$

$$= \frac{\alpha^2 r_1^{(m+1)^n+(m-1)^n} + \beta^2 r_2^{(m-1)^n+(m-1)^n}}{\left(\alpha r_1^{m^n} - \beta r_2^{m^n}\right)^2}$$

$$-\alpha\beta\left(r_1 r_2\right)^{(m-1)^n} \frac{\left(r_1^{(m+1)^n-(m-1)^n} + r_2^{(m+1)^n-(m-1)^n}\right)}{\left(\alpha r_1^{m^n} - \beta r_2^{m^n}\right)^2}$$

$$= \frac{\alpha^2 r_1^{(m+1)^n+(m-1)^n} + \beta^2 r_2^{(m-1)^n+(m-1)^n}}{\left(\alpha r_1^{m^n} - \beta r_2^{m^n}\right)^2}$$

$$-\alpha\beta\left(-1\right)^{(m-1)^n} \cdot \frac{\left(r_1^{(m+1)^n-(m-1)^n} + r_2^{(m+1)^n-(m-1)^n}\right)}{\left(\alpha r_1^{m^n} - \beta r_2^{m^n}\right)^2}$$

$$= \frac{\alpha^2 r_1^{(m+1)^n+(m-1)^n} - \alpha\beta\left(-1\right)^{(m-1)^n} r_1^{(m+1)^n-(m-1)^n}}{\left(\alpha r_1^{m^n} - \beta r_2^{m^n}\right)^2}$$

$$+ \frac{\beta^2 r_2^{(m-1)^n+(m-1)^n} - \alpha\beta\left(-1\right)^{(m-1)^n} r_2^{(m+1)^n-(m-1)^n}}{\left(\alpha r_1^{m^n} - \beta r_2^{m^n}\right)^2}$$

$$= \frac{\alpha^2 r_1^{(m+1)^n+(m-1)^n-2m^n} - \alpha\beta(-1)^{(m-1)^n} r_1^{(m+1)^n-(m-1)^n-2m^2}}{\left(\alpha-\beta\left(\frac{r_2}{r_1}\right)^{m^n}\right)^2} + \frac{\beta^2 r_2^{(m+1)^n+(m-1)^n-2m^n} - \alpha\beta(-1)^{(m-1)^n} r_2^{(m+1)^n-(m-1)^n-2m^2}}{\left(\alpha\left(\frac{r_1}{r_2}\right)^{m^n}-\beta\right)^2}. \quad (5)$$

We have

$$\lim_{m\to\infty} \frac{\alpha^2 r_1^{(m+1)^n+(m-1)^n-2m^n} - \alpha\beta(-1)^{(m-1)^n} r_1^{(m+1)^n-(m-1)^n-2m^2}}{\left(\alpha-\beta\left(\frac{r_2}{r_1}\right)^{m^n}\right)^2} = \begin{cases} 0, & if\ n=1, \\ r_1^2, & if\ n=2, \\ \infty, & if\ n=3. \end{cases}$$

Since $0 < |r_2| < 1$ and $|r_2| < |r_1|$, , it follows that

$$\lim_{m\to\infty} \frac{\beta^2 r_2^{(m+1)^n+(m-1)^n-2m^n} - \alpha\beta(-1)^{(m-1)^n} r_2^{(m+1)^n-(m-1)^n-2m^2}}{\left(\alpha\left(\frac{r_1}{r_2}\right)^{m^n}-\beta\right)^2} = 0.$$

Therefore,

$$\frac{a_{m+1}^{(n)}}{a_m^{(n)}} = \begin{cases} 0, & if\ n=1, \\ r_1^2, & if\ n=2, \\ \infty, & otherwise. \end{cases}$$

By (5), we can also conclude that

$$\frac{a_{(m+1)^{(n)}}}{a_m^{(n)}} \in O(r_1^{2m^{n-2}}).$$

Theorem 2.1 implies that, for any positive integer k , the growth rate of the ratios of consecutive terms of the $n-$ exponential subsequence $\{a_{m^n}\}_{m=0}^{\infty}$ converges if and only if $n \le 2$.

Theorem 2.1 can be generalized to the sequences $\{b_m\}_{m=0}^{\infty}$ defined by

$$b_m = k_1 b_{m-1} + k_2 b_{m-2}, \text{ for } m \ge 2 \quad (6)$$

where k_1, k_2 are non-negative integers and $b_0 = s$, $b_1 = t$. If the roots of the characteristic equation of (6) are distinct, then the Binet formula of b_m is

$$b_m = \frac{1}{l_1 - l_2}\left((t - s l_2) l_1^m + (s l_1 - t) l_2^m\right), \quad (7)$$

where

$$l_1 = \frac{k_1 + \sqrt{k_1^2 + 4k_2}}{2}$$

and

$$l_2 = \frac{k_1 - \sqrt{k_1^2 + 4k_2}}{2} .$$

If $1 - k_1 < k_2 < 0,$ then $0 < l_2 < 1$ and $l_2 < l_1.$

So, we are able to extend the same method appearing in Theorem 2.1 to Theorem 2.2.

Theorem 2.2. If b_{m^n} is not zero for all $m, n \ge 1$ and $1 - k_1 < k_2 < 0,$ in (6), then

$$\frac{b_{m+1}^{(n)}}{b_m^{(n)}} = \begin{cases} 0, & if\ n=1, \\ l_1^2, & if\ n=2, \\ \infty, & otherwise. \end{cases}$$

As a result, the quotient of the ratios of the $n-$ exponential subsequence of the Fibonacci sequences converges to the square of the golden ratio.

Let F_m, L_m, P_m, Q_m be the Fibonacci number, Lucas Number, Pell number and Pell-Lucas number, respectively. Let φ be the golden ratio and $\delta = 1 + \sqrt{2}$.

Corollary 2.3. The following statements are true:

- $\lim_{m\to\infty} \frac{F_m^{(2)}}{F_{m-1}^{(2)}} = \varphi^2$
- $\lim_{m\to\infty} \frac{L_m^{(2)}}{L_{m-1}^{(2)}} = \varphi^2$
- $\lim_{m\to\infty} \frac{P_m^{(2)}}{P_{m-1}^{(2)}} = \delta^2$
- $\lim_{m\to\infty} \frac{Q_m^{(2)}}{Q_{m-1}^{(2)}} = \delta^2.$

Corollary 2.4.

$$\lim_{m\to\infty} \frac{F_{k,m}^{(2)}}{F_{k,m-1}^{(2)}} = \begin{cases} \frac{\left(k+\sqrt{k^2+4}\right)^2}{4}, & if\ k>0, \\ \frac{\left(k-\sqrt{k^2+4}\right)^2}{4}, & if\ k<0. \end{cases}$$

Example 2.5 gives an example of the sequences satisfying Theorem 2.2 but not the sequences in Corollary 2.3 and 2.4.

Example 2.5. Let $b_m = 3b_{m-1} + 2b_{m-2}$, , where $b_0 = 0, b_1 = 1.$. Table 1 shows the value of b_{m^2}, for

$m = 1,\ldots,10.$ By Theorem 2.2,

$$\lim_{m\to\infty} \frac{b_{(m+1)^2} b_{(m-1)^2}}{b_{m^2}} = \frac{\left(3+\sqrt{17}\right)^2}{4}.$$

Table 1: b_{m^2}

b_{m^2}	value of b_{m^2}
b_1	1
b_4	39
b_9	22363
b_{16}	162557031
b_{25}	14988571946011
b_{36}	17530468900008685335
b_{49}	260079179143778066525568571
b_{64}	48943657027144499564640559765030311
b_{81}	116833133373681561419044674956313653328090043
b_{100}	3537646303459605111696665428274832196761996930395731479

Discussion

Theorem 2.2 implies that all sequences satisfying the recurrence relation (6) with a condition that $1-k_1 < k_2 < 0,$ and b_{m^n} is non-null real number, for $m,n \geq 1$. The growth of the ratios of consecutive terms of the subsequence $\left\{b_{m^n}\right\}$ is $O\left(l_1^{2m^{n-2}}\right)$. It converges if and only if $n \leq 2$. Moreover, if $n = 2$, then

$$\lim_{m\to\infty} \frac{b_{(m+1)^2} b_{(m-1)^2}}{b_{m^2}} = \frac{2k_1^2 + 2\sqrt{k_1^2+4k_2} + 4k_2}{4}.$$

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