การกำกับที่เป็นดีกรี-เมจิกบนการดำเนินการทวิภาคของกราฟสองส่วนแบบบริบูรณ์และกราฟสามส่วนแบบบริบูรณ์

Degree-Magic Labelings on Binary Operations of Complete Bipartite and Tripartite Graphs

ไภสัชชา อินพูลใจ[1]

Phaisatcha Inpoonjai[1]

Received: 19 March 2018; Accepted: 1 June 2018

บทคัดย่อ

กราฟจะถูกเรียกว่า ซุปเปอร์เมจิก ถ้ามีการกำกับของเส้นด้วยจำนวนเต็มบวกที่แตกต่างและเรียงต่อกัน ซึ่งผลรวมของตัวเลขของทุกเส้นที่เชื่อมกับจุดใด ๆ เป็นค่าคงตัว กราฟ G จะถูกเรียกว่า ดีกรี-เมจิก ถ้ามีการกำกับของเส้นด้วยจำนวนเต็ม $1,2,\dots,|E(G)|$ ซึ่งผลรวมของตัวเลขของเส้นที่เชื่อมกับจุด v ใด ๆ เท่ากับ $(1+|E(G)|)\deg(v)/2$ กราฟดีกรี-เมจิกขยายกราฟปรกติซุปเปอร์เมจิก ในงานวิจัยนี้ มีการพิสูจน์เงื่อนไขที่จำเป็นและเพียงพอสำหรับการมีอยู่ของการกำกับที่เป็นดีกรี-เมจิกของกราฟภายใต้การดำเนินการทวิภาคของกราฟสองส่วนแบบบริบูรณ์และกราฟสามส่วนแบบบริบูรณ์

คำสำคัญ: Supermagic graphs, Degree-magic graphs, Balanced degree-magic graphs.

Abstract

A graph is called supermagic if there is a labeling of edges where all edges are differently labeled with consecutive positive integers such that the sum of the labels of all edges which are incident to each vertex of this graph is a constant. A graph G is called degree-magic if all edges can be labeled by integers $1,2,\dots,|E(G)|$ so that the sum of the labels of the edges which are incident to any vertex v is equal to $(1+|E(G)|)\deg(v)/2$. Degree-magic graphs extend supermagic regular graphs. In this paper, the necessary and sufficient conditions for the existence of degree-magic labelings of graphs under binary operations of complete bipartite and tripartite graphs are proved.

Keywords: Supermagic graphs, Degree-magic graphs, Balanced degree-magic graphs.

Introduction

One considers simple graphs without isolated vertices. If G is a graph, then $V(G)$ and $E(G)$ stand for the vertex set and the edge set of G, respectively. Cardinalities of these sets are called the *order* and *size* of G.

Let a graph G and a mapping f from $E(G)$ into the set of positive integers be given. The *index mapping* of f is the mapping f^* from $V(G)$ into positive integers defined by

$$f^*(v) = \sum_{e \in E(G)} \eta(v,e) f(e) \text{ for every } v \in V(G), \tag{1.1}$$

where $\eta(v,e)$ is equal to 1 when e is an edge incident with a vertex v, and 0 otherwise. An injective mapping f from $E(G)$ into the set of positive integers is called a *magic labeling* of G for an *index* λ if its index mapping f^* satisfies $f^*(v) = \lambda$ for all $v \in V(G)$. A magic labeling f of a graph G is called a *supermagic labeling* if the set $\{f(e) : e \in E(G)\}$ consists of consecutive positive integers. A graph G is *supermagic* (*magic*) whenever a supermagic (magic) labeling of G exists.

[1] คณะวิทยาศาสตร์และเทคโนโลยีการเกษตร มหาวิทยาลัยเทคโนโลยีราชมงคลล้านนา เชียงราย 99 หมู่ 10 ตำบลทรายขาว อำเภอพาน จังหวัดเชียงราย 57120

[1] Faculty of Sciences and Agricultural Technology, Rajamangala University of Technology Lanna Chiang Rai 99, Moo 10, Sai Khao Sub District, Phan District, Chiang Rai Province, 57120, Thailand phaisatcha_in@outlook.com

A bijective mapping f from $E(G)$ into the set $\{1,2,\dots,|E(G)|\}$ is called a *degree-magic labeling* (or only *d-magic labeling*) of G if its index mapping f^* satisfies $f^*(v) = \frac{1+|E(G)|}{2}\deg(v)$ for all $v \in V(G)$.

A degree-magic labeling f of a graph G is called *balanced* if for all $v \in V(G)$, the following equation is satisfied

$$\left|\left\{e \in E(G): \eta(v,e) = 1, f(e) \le \left\lfloor\frac{|E(G)|}{2}\right\rfloor\right\}\right| = \left|\left\{e \in E(G): \eta(v,e) = 1, f(e) > \left\lfloor\frac{|E(G)|}{2}\right\rfloor\right\}\right|.$$

One says that a graph G is *degree-magic* (*balanced degree-magic*) or only *d-magic* when a d-magic (balanced d-magic) labeling of G exists.

A graph G is a *bipartite graph* if $V(G)$ can be partitioned into two disjoint subsets U and W, called *partite sets*, such that every edge of G joins a vertex of U and a vertex of W. If every vertex of U is adjacent to every vertex of W, then G is a *complete bipartite graph*. A graph G is called *k-partite graph* if $V(G)$ can be partitioned into k disjoint subsets $V_1, V_2, \dots, V_k$, once again called *partite sets*, such that uv is an edge of G if u and v belong to different partite sets. If every two vertices in different partite sets are joined by an edge, then G is a *complete k-partite graph*. For any graph G, the *graph union* of two graphs G, denoted by $G \cup G$ or $2G$, is a graph whose vertex set and edge set are the disjoint unions of the vertex sets and edge sets of two graphs G, respectively. For any two vertex-disjoint graphs G and H, the *join* of graphs G and H, denoted by $G + H$, consists of $G \cup H$ and all edges joining a vertex of G and a vertex of H. The *composition* of graphs G and H, denoted by $G \cdot H$, is a graph such that the vertex set of $G \cdot H$ is the Cartesian product $V(G) \times V(H)$ and any two vertices (u,v) and (x,y) are adjacent in $G \cdot H$ if and only if either u is adjacent to x in G or $u = x$ and v is adjacent to y in H. The *Cartesian product* of graphs G and H, denoted by $G \times H$, is a graph such that the vertex set of $G \times H$ is the Cartesian product $V(G) \times V(H)$ and any two vertices (u,v) and (x,y) are adjacent in $G \times H$ if and only if either $u = x$ and v is adjacent to y in H or $v = y$ and u is adjacent to x in G. The *tensor product* of graphs G and H, denoted by $G \oplus H$, is a graph such that the vertex set of $G \oplus H$ is the Cartesian product $V(G) \times V(H)$ and any two vertices (u,v) and (x,y) are adjacent in $G \oplus H$ if and only if u is adjacent to x in G and v is adjacent to y in H.

The concept of magic graphs was introduced by Sedláček[1]. Later, supermagic graphs were introduced by Stewart[2]. There are now many papers published on magic and supermagic graphs; see[3-5] for more comprehensive references. The concept of degree-magic graphs was then introduced by Bezegová and Ivančo[6] as an extension of supermagic regular graphs. They established the basic properties of degree-magic graphs and characterized degree-magic and balanced degree-magic complete bipartite graphs in[6]. They also characterized degree-magic complete tripartite graphs in[7]. Some of these concepts are investigated in[8-10]. One will hereinafter use the auxiliary results from these studies.

Theorem 1.1[6] *Let G be a regular graph. Then G is supermagic if and only if it is d-magic.*

Theorem 1.2[6] *Let G be a d-magic graph of even size. Then every vertex of G has an even degree and every component of G has an even size.*

Theorem 1.3[6] *Let G be a balanced d-magic graph. Then G has an even number of edges and every vertex has an even degree.*

Theorem1.4[6] *Let G be a d-magic graph having a half-factor. Then $2G$ is a balanced d-magic graph.*

Theorem 1.5[6] *Let H_1 and H_2 be edge-disjoint subgraphs of a graph G which form its decomposition. If H_1 is d-magic and H_2 is balanced d-magic, then G is a d-magic graph. Moreover, if H_1 and H_2 are both balanced d-magic, then G is a balanced d-magic graph.*

Proposition 1.6[6] *For $p, q > 1$, the complete bipartite graph $K_{p,q}$ is d-magic if and only if $p \equiv q \pmod 2$ and $(p,q) \neq (2,2)$.*

Theorem 1.7[6] *The complete bipartite graph $K_{p,q}$ is balanced d-magic if and only if the following statements hold:*
$p \equiv q \equiv 0 \pmod 2$;

if $p \equiv q \equiv 2 \pmod 4$, then $\min\{p,q\} \ge 6$.

Lemma 1.8[7]] *Let p, q and r be even positive integers. Then the complete tripartite graph $K_{p,q,r}$ is balanced d-magic.*

Lemma 1.9[7] *Let q and r be odd positive integers with $q \geq r$ and let p be an even positive integer such that $p \equiv 0 (\mathrm{mod}\ 4)$ whenever $q = 1$. Then the complete tripartite graph $K_{p,q,r}$ is d-magic.*

Labelings in the Join of Complete Bipartite and Tripartite Graphs

For any positive integers p, q, r, s and t, one considers the join $K_{p,q} + K_{r,s,t}$ of complete bipartite and tripartite graphs. Let $K_{p,q} + K_{r,s,t}$ be a d-magic graph. Since $\deg(v)$ is $p + r + s + t$, $q + r + s + t$, $p + q + r + s$, $p + q + r + t$ or $p + q + s + t$ and

$$f^*(v) = (pq + rs + rt + st + (p + q)(r + s + t) + 1)\deg(v)/2$$

for any vertex $v \in V(K_{p,q} + K_{r,s,t})$, one has the following proposition.

Proposition 2.1 *Let $K_{p,q} + K_{r,s,t}$ be a d-magic graph. Then the following statements hold:*

only two of p, q, r, s and t are even or

only three of p, q, r, s and t are even or

all of p, q, r, s and t are either odd or even.

Proof. Assume that f is a d-magic labeling of $K_{p,q} + K_{r,s,t}$. Suppose to the contrary that only one of p, q, r, s and t is either odd or even. Thus, $p + r + s + t$, $q + r + s + t$, $p + q + r + s$, $p + q + r + t$ or $p + q + s + t$ is odd, and $pq + rs + rt + st + (p + q)(r + s + t) + 1$ is odd. Since f satisfies

$f^*(v) = (pq + rs + rt + st + (p + q)(r + s + t) + 1)\deg(v)/2$ and it is not an integer for some vertex $v \in V(K_{p,q} + K_{r,s,t})$, by (1.1), $f^*(v)$ is a sum of integers, one has a contradiction.

Proposition 2.2 *Let $K_{p,q} + K_{r,s,t}$ be a balanced d-magic graph. Then p, q, r, s and t are either odd or even.*

Proof. Suppose to the contrary that some of p, q, r, s and t are odd and some are even. Thus, $p + r + s + t$, $q + r + s + t$, $p + q + r + s$, $p + q + r + t$ or $p + q + s + t$ is odd. This means that some vertices of $K_{p,q} + K_{r,s,t}$ have odd degrees. Since every vertex of balanced d-magic graph has an even degree, one has a contradiction.

In the next result, one shows sufficient conditions for the existence of d-magic labelings of the join of complete bipartite and tripartite graphs $K_{p,q} + K_{r,s,t}$.

Proposition 2.3 *Let p and q be even positive integers, let s and t be odd positive integers with $s \geq t$ and let r be an even positive integer such that $r \equiv 0 (\mathrm{mod}\ 4)$ whenever $s = 1$. Then $K_{p,q} + K_{r,s,t}$ is a d-magic graph.*

Proof. Let p and q be even positive integers, let s and t be odd positive integers with $s \geq t$ and let r be an even positive integer such that $r \equiv 0 (\mathrm{mod}\ 4)$ whenever $s = 1$. Then the graph $K_{r,s,t}$ is d-magic by Lemma 1.9. Since p, q and $r + s + t$ are even, $K_{p,q,r+s+t}$ is balanced d-magic by Lemma 1.8. Since $K_{p,q} + K_{r,s,t}$ is the graph such that $K_{r,s,t}$ and $K_{p,q,r+s+t}$ form its decomposition, $K_{p,q} + K_{r,s,t}$ is a d-magic graph by Theorem 1.5.

Proposition 2.4 *Let p, q, r, s and t be even positive integers. Then $K_{p,q} + K_{r,s,t}$ is a balanced d-magic graph.*

Proof. Let p, q, r, s and t be even positive integers. Then the graphs $K_{r,s,t}$ and $K_{p,q,r+s+t}$ are balanced d-magic by Lemma 1.8. Since $K_{p,q} + K_{r,s,t}$ is the graph such that $K_{r,s,t}$ and $K_{p,q,r+s+t}$ form its decomposition, $K_{p,q} + K_{r,s,t}$ is a balanced d-magic graph by Theorem 1.5.

Corollary 2.5 *Let p, q, r, s and t be even positive integers. If $p = q = r = s = t$, then $K_{p,q} + K_{r,s,t}$ is a supermagic graph.*

Proof. This follows from Theorem 1.1 and Proposition 2.4.

Example 2.1 One considers the join of complete bipartite and tripartite graphs $K_{2,2}$ and $K_{2,2,4}$. A balanced d-magic graph $K_{2,2} + K_{2,2,4}$ is constructed (see Figure 1) and the labels on edges of $K_{2,2} + K_{2,2,4}$ are shown in Table 1

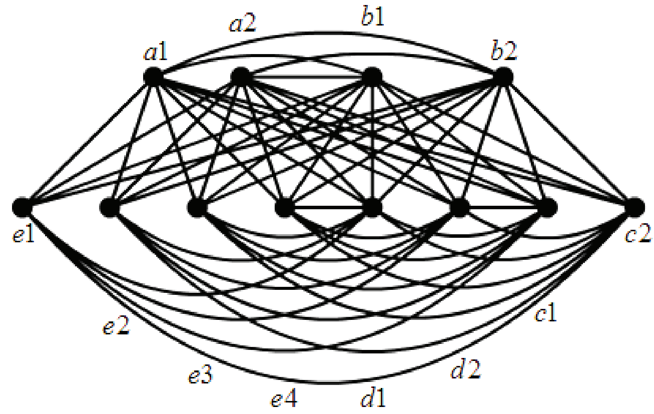

Figure 1 A balanced d-magic graph $K_{2,2}+K_{2,2,4}$.

Table 1 The labels on edges of balanced d-magic graph $K_{2,2}+K_{2,2,4}$.

vertex	c1	c2	d1	d2	e1	e2	e3	e4	b1	b2	vertex	d1	d2	e1	e2	e3	e4
a1	51	6	7	50	11	46	15	42	53	4	c1	19	22	37	36	28	29
a2	5	52	49	8	45	12	41	16	56	1	c2	38	35	21	20	30	27
b1	3	2	48	9	44	13	40	17	-	-	d1	-	-	23	34	31	26
b2	55	54	10	47	14	43	18	39	-	-	d2	-	-	33	24	25	32

Labelings in the Composition of Complete Bipartite and Tripartite Graphs

For any positive integers p,q,r,s and t, one considers the composition $K_{p,q}\cdot K_{r,s,t}$ of complete bipartite and tripartite graphs. Let $K_{p,q}\cdot K_{r,s,t}$ be a d-magic graph. Since $\deg(v)$ is $(r+s+t)p+r+s$, $(r+s+t)p+r+t$, $(r+s+t)p+s+t$, $(r+s+t)q+r+s$, $(r+s+t)q+r+t$ or $(r+s+t)q+s+t$ and $f^*(v)=\left((r+s+t)^2pq+(rs+rt+st)(p+q)+1\right)\deg(v)/2$ for any vertex $v\in V\left(K_{p,q}\cdot K_{r,s,t}\right)$, one has the following proposition.

Proposition 3.1 *Let $K_{p,q}\cdot K_{r,s,t}$ be a d-magic graph. Then the following statements hold:*

p or q is odd and r,s and t are even or

only one of p and q is even and r,s and t are odd or

only two of p,q,r,s and t are even or

all of p,q,r,s and t are either odd or even.

Proof. Assume that f is a d-magic labeling of $K_{p,q}\cdot K_{r,s,t}$. Suppose to the contrary that only one of r,s and t is odd and p and q are both even, only one of r,s and t is even and p and q are both odd or only three of p,q,r,s and t are even and p or q is even. Thus, $(r+s+t)p+r+s$, $(r+s+t)p+r+t$, $(r+s+t)p+s+t$, $(r+s+t)q+r+s$, $(r+s+t)q+r+t$ or $(r+s+t)q+s+t$ is odd and $(r+s+t)^2pq+(rs+rt+st)(p+q)+1$ is odd. Since the mapping f satisfies $f^*(v)=\left((r+s+t)^2pq+(rs+rt+st)(p+q)+1\right)\deg(v)/2$ and it is not an integer for some vertex $v\in V\left(K_{p,q}\cdot K_{r,s,t}\right)$, by (1.1), $f^*(v)$ is a sum of integers, one has a contradiction.

Proposition 3.2 *Let $K_{p,q}\cdot K_{r,s,t}$ be a balanced d-magic graph. Then the following statements hold:*

p or q is odd and r,s and t are even or

p and q are even and r,s and t are odd or

p,q,r,s and t are even.

Proof. Suppose to the contrary that all of p,q,r,s and t are odd, only one of p,q,r,s and t is even, only two of p,q,r,s and t are even and p or q is odd, only three of p,q,r,s and t are even and p or q is even or only one of r,s and t is odd and p and q are both even. Thus, $(r+s+t)p+r+s$, $(r+s+t)p+r+t$, $(r+s+t)p+s+t$, $(r+s+t)q+r+s$, $(r+s+t)q+r+t$ or $(r+s+t)q+s+t$ is odd.

This means that some vertices of $K_{p,q} \cdot K_{r,s,t}$ have odd degrees. Since every vertex of balanced d-magic graph has an even degree, one has a contradiction.

In the next result, one is able to find a sufficient condition for the existence of d-magic labelings of the composition of complete bipartite and tripartite graphs $K_{p,q} \cdot K_{r,s,t}$.

Proposition 3.3 *Let p and q be positive integers and let r, s and t be even positive integers. Then $K_{p,q} \cdot K_{r,s,t}$ is a balanced d-magic graph.*

Proof. Let p and q be positive integers and let r, s and t be even positive integers. Since $r + s + t \geq 6$ and it is even, the graph $K_{r+s+t,r+s+t}$ is balanced d-magic by Theorem 1.7. The graph $K_{r,s,t}$ is balanced d-magic by Lemma 1.8. The graph $K_{p,q} \cdot K_{r,s,t}$ is decomposable into pq balanced d-magic subgraphs isomorphic to $K_{r+s+t,r+s+t}$ and $p+q$ balanced d-magic subgraphs isomorphic to $K_{r,s,t}$. According to Theorem 1.5, $K_{p,q} \cdot K_{r,s,t}$ is a balanced d-magic graph.

Corollary 3.4 *Let p and q be positive integers and let r, s and t be even positive integers. If $p = q$ and $r = s = t$, then $K_{p,q} \cdot K_{r,s,t}$ is a supermagic graph.*

Proof. This follows from Theorem 1.1 and Proposition 3.3.

Example 3.1 One considers the composition of complete bipartite and tripartite graphs $K_{1,2}$ and $K_{2,2,2}$. A balanced d-magic graph $K_{1,2} \cdot K_{2,2,2}$ is constructed (see Figure 2) with the labels on edges of $K_{1,2} \cdot K_{2,2,2}$ in Table 2

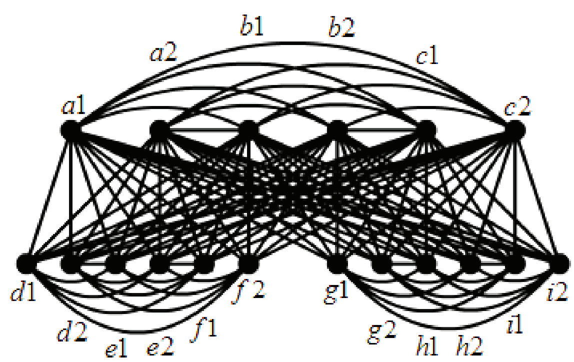

Figure 2 A balanced d-magic graph $K_{1,2} \cdot K_{2,2,2}$.

Table 2 The labels on edges of balanced d-magic graph $K_{1,2} \cdot K_{2,2,2}$.

vertex	d1	d2	e1	e2	f1	f2	g1	g2	h1	h2	i1	i2
a1	19	84	73	36	30	85	1	102	91	18	12	103
a2	89	26	77	35	80	20	107	8	95	17	98	2
b1	22	82	33	75	27	88	4	100	15	93	9	106
b2	87	81	34	76	28	21	105	99	16	94	10	3
c1	86	29	32	74	83	23	104	11	14	92	101	5
c2	24	25	78	31	79	90	6	7	96	13	97	108

vertex	e1	e2	f1	f2	vertex	h1	h2	i1	i2	vertex	b1	b2	c1	c2
d1	43	46	65	64	**g1**	37	40	71	70	**a1**	54	56	50	58
d2	66	63	45	44	**g2**	72	69	39	38	**a2**	55	53	51	59
e1	-	-	47	62	**h1**	-	-	41	68	**b1**	-	-	57	52
e2	-	-	61	48	**h2**	-	-	67	42	**b2**	-	-	60	49

Labelings in the Cartesian Product of Complete Bipartite and Tripartite Graphs

For any positive integers p, q, r, s and t, one considers the Cartesian product $K_{p,q} \times K_{r,s,t}$ of complete bipartite and tripartite graphs. Let $K_{p,q} \times K_{r,s,t}$ be a d-magic graph. Since $\deg(v)$ is $p+r+s$, $p+r+t$, $p+s+t$, $q+r+s$, $q+r+t$ or $q+s+t$ and $f^*(v) = ((p+q)(rs+rt+st) + pq(r+s+t) + 1)\deg(v)/2$ for any vertex $v \in V(K_{p,q} \times K_{r,s,t})$, one has the following proposition.

Proposition 4.1 *Let $K_{p,q} \times K_{r,s,t}$ be a d-magic graph. Then the following statements hold:*

only two of p, q, r, s and t are even or

only one of p and q is even and r, s and t are odd or

all of p, q, r, s and t are either odd or even.

Proof. Assume that f is a d-magic labeling of $K_{p,q} \times K_{r,s,t}$. Suppose to the contrary that only four of p, q, r, s and t are even, only three of p, q, r, s and t are even or only one of r, s and t is even and p and q are both odd. Therefore, $p+r+s$, $p+r+t$, $p+s+t$, $q+r+s$, $q+r+t$ or $q+s+t$ is odd and $(p+q)(rs+rt+st) + pq(r+s+t) + 1$ is odd. Since the mapping f^* satisfies $f^*(v) = ((p+q)(rs+rt+st) + pq(r+s+t) + 1)\deg(v)/2$ and it is not an integer for some vertex $v \in V(K_{p,q} \times K_{r,s,t})$, by (1.1), $f^*(v)$ is a sum of integers, one has a contradiction.

Proposition 4.2 *Let $K_{p,q} \times K_{r,s,t}$ be a balanced d-magic graph. Then the following statements hold:*

p and q are even and r, s and t are odd or

p, q, r, s and t are even.

Proof. Suppose to the contrary that only four of p, q, r, s and t are even, only three of p, q, r, s and t are even, only two of p, q, r, s and t are even and p or q is odd, only one of p, q, r, s and t is even or all of p, q, r, s and t are odd. Thus, $p+r+s$, $p+r+t$, $p+s+t$, $q+r+s$, $q+r+t$ or $q+s+t$ is odd. This means that some vertices of $K_{p,q} \times K_{r,s,t}$ have odd degrees. Since every vertex of balanced d-magic graph has an even degree, one has a contradiction.

In the next result, one finds a sufficient condition for the existence of d-magic labelings of the Cartesian product of complete bipartite and tripartite graphs $K_{p,q} \times K_{r,s,t}$.

Proposition 4.3 *Let p, q, r, s and t be even positive integers and $(p,q) \neq (2,2)$. Then $K_{p,q} \times K_{r,s,t}$ is a balanced d-magic graph.*

Proof. Let p, q, r, s and t be even positive integers and $(p,q) \neq (2,2)$. Since the graph $K_{p,q}$ is d-magic by Proposition 1.6, $2K_{p,q}$ is a balanced d-magic graph by Theorem 1.4. The graph $K_{r,s,t}$ is balanced d-magic by Lemma 1.8. The graph $K_{p,q} \times K_{r,s,t}$ is decomposable into $(r+s+t)/2$ balanced d-magic subgraphs isomorphic to $2K_{p,q}$ and $p+q$ balanced d-magic subgraphs isomorphic to $K_{r,s,t}$. According to Theorem 1.5, $K_{p,q} \times K_{r,s,t}$ is a balanced d-magic graph.

Corollary 4.4 *Let p, q, r, s and t be even positive integers and $(p,q) \neq (2,2)$. If $p = q$ and $r = s = t$, then $K_{p,q} \times K_{r,s,t}$ is a supermagic graph.*

Proof. This follows from Theorem 1.1 and Proposition 4.3.

Example 4.1 One considers the Cartesian product of complete bipartite and tripartite graphs $K_{2,4}$ and $K_{2,2,2}$. A balanced d-magic graph $K_{2,4} \times K_{2,2,2}$ is constructed (see Figure 3) and the labels on edges of $K_{2,4} \times K_{2,2,2}$ are shown in Table 3

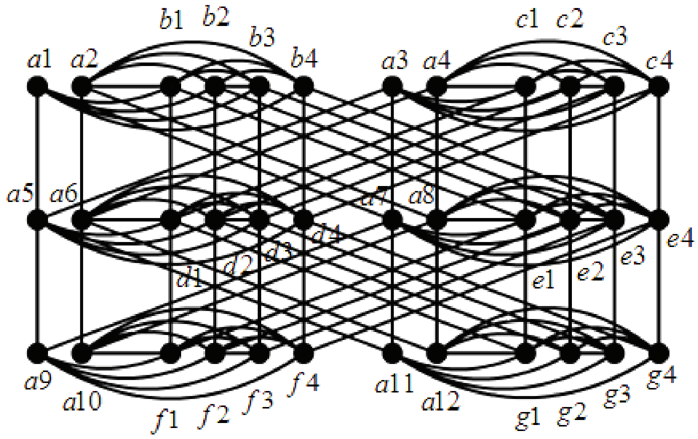

Figure 3 A balanced d-magic graph $K_{2,4} \times K_{2,2,2}$.

Table 3 The labels on edges of balance d-magic graph $K_{2,4} \times K_{2,2,2}$.

vertex	a9	a10	d1	d2	d3	d4	e1	e2	e3	e4	f3	f4
f1	95	94	71	-	-	-	50	-	-	-	25	28
f2	27	26	-	75	-	-	-	46	-	-	96	93
f3	29	92	-	-	63	-	-	-	58	-	-	-
f4	91	30	-	-	-	67	-	-	-	54	-	-

vertex	a11	a12	d1	d2	d3	d4	e1	e2	e3	e4	g3	g4
g1	89	88	49	-	-	-	72	-	-	-	31	34
g2	33	32	-	45	-	-	-	76	-	-	90	87
g3	35	86	-	-	57	-	-	-	64	-	-	-
g4	85	36	-	-	-	53	-	-	-	68	-	-

vertex	d1	d2	d3	d4	a1	a2	a3	a4	vertex	e1	e2	e3	e4	a1	a2	a3	a4
a5	107	15	17	103	44	-	78	-	**a7**	101	21	23	97	77	-	43	-
a6	106	14	104	18	-	40	-	82	**a8**	100	20	98	24	-	81	-	39

vertex	d1	d2	d3	d4	e1	e2	e3	e4	a1	a2	b3	b4	vertex	d1	d2	e1	e2
b1	52	-	-	-	69	-	-	-	119	118	1	4	**d3**	13	108	-	-
b2	-	48	-	-	-	73	-	-	3	2	120	117	**d4**	16	105	-	-
b3	-	-	60	-	-	-	61		5	116	-	-	**e3**	-	-	19	102
b4	-	-	-	56	-	-	-	65	115	6	-	-	**e4**	-	-	22	99

c1	70	-	-	-	51	-	-	-	113	112	7	10	**a5**	79	-	41	-
c2	-	74	-	-	-	47	-	-	9	8	114	111	**a6**	-	83	-	37
c3	-	-	62	-	-	-	59	-	11	110	-	-	**a7**	42	-	80	-
c4	-	-	-	66	-	-	-	55	109	12	-	-	**a8**	-	38	-	84

Labelings in the Tensor Product of Complete Bipartite and Tripartite Graphs

For any positive integers p, q, r, s and t, one considers the tensor product $K_{p,q} \oplus K_{r,s,t}$ of complete bipartite and tripartite graphs. Let $K_{p,q} \oplus K_{r,s,t}$ be a d-magic graph. Since $\deg(v)$ is $p(r+s)$, $p(r+t)$, $p(s+t)$, $q(r+s)$, $q(r+t)$ or $q(s+t)$ and $f^*(v) = (2pq(rs+rt+st)+1)\deg(v)/2$ for any vertex $v \in V(K_{p,q} \oplus K_{r,s,t})$, one has the following proposition.

Proposition 5.1 *Let $K_{p,q} \oplus K_{r,s,t}$ be a d-magic graph. Then the following statements hold:*

only four of p, q, r, s and t are even or

only three of p, q, r, s and t are even and p and q are even or

p and q are odd and r, s and t are even or

p or q is even and r, s and t are odd or

all of p, q, r, s and t are either odd or even.

Proof. Assume that f is a d-magic labeling of $K_{p,q} \oplus K_{r,s,t}$. Suppose to the contrary that only three of p, q, r, s and t are even and only two of r, s and t are even, only two of p, q, r, s and t are even and p or q is odd or only one of r, s and t is even and p and q are both odd. Thus, $p(r+s)$, $p(r+t)$, $p(s+t)$, $q(r+s)$, $q(r+t)$ or $q(s+t)$ is odd and $2pq(rs+rt+st)+1$ is odd. Since f satisfies $f^*(v) = (2pq(rs+rt+st)+1)\deg(v)/2$ and it is not an integer for some vertex $v \in V(K_{p,q} \oplus K_{r,s,t})$, by (1.1), $f^*(v)$ is a sum of integers, one has a contradiction.

Proposition 5.2 *Let $K_{p,q} \oplus K_{r,s,t}$ be a balanced d-magic graph. Then the following statements hold:*

only four of p, q, r, s and t are even or

only three of p, q, r, s and t are even and p and q are even or

p and q are odd and r, s and t are even or

p or q is even and r, s and t are odd or

all of p, q, r, s and t are either odd or even.

Proof. Suppose to the contrary that only three of p, q, r, s and t are even and only two of r, s and t are even, only two of p, q, r, s and t are even and p or q is odd or only one of r, s and t is even and p and q are both odd. Thus, $p(r+s)$, $p(r+t)$, $p(s+t)$, $q(r+s)$, $q(r+t)$ or $q(s+t)$ is odd. This means that some vertices of $K_{p,q} \oplus K_{r,s,t}$ have odd degrees. Since every vertex of balanced d-magic graph has an even degree, one has a contradiction.

In the next result, one finds a sufficient condition for the existence of d-magic labelings of the tensor product of complete bipartite and tripartite graphs $K_{p,q} \oplus K_{r,s,t}$.

Proposition 5.3 *Let p or q be even positive integers and let r, s and t be even positive integers. Then $K_{p,q} \oplus K_{r,s,t}$ is a balanced d-magic graph.*

Proof. Let p or q be even positive integers and let r, s and t be even positive integers. One considers the following two cases:

Case I. If q is even. Then $(s+t)q$, $(r+t)q$ and $(r+s)q$ are not congruent to 2 modulo 4. Thus, the graph $K_{r,(s+t)q}$, $K_{s,(r+t)q}$ and $K_{t,(r+s)q}$ are balanced d-magic by Theorem 1.7. The graph $K_{p,q} \oplus K_{r,s,t}$ is decomposable into p balanced d-magic subgraphs isomorphic to $K_{r,(s+t)q}$, $K_{s,(r+t)q}$ and $K_{t,(r+s)q}$. According to Theorem 1.5, $K_{p,q} \oplus K_{r,s,t}$ is a balanced d-magic graph.

Case II. If p is even. Then $(s+t)p$, $(r+t)p$ and $(r+s)p$ are not congruent to 2 modulo 4. Thus, the graph $K_{r,(s+t)p}$, $K_{s,(r+t)p}$ and $K_{t,(r+s)p}$ are balanced d-magic by Theorem 1.7. The graph $K_{p,q} \oplus K_{r,s,t}$ is decomposable into q balanced d-magic subgraphs isomorphic to $K_{r,(s+t)p}$, $K_{s,(r+t)p}$ and $K_{t,(r+s)p}$. According to Theorem 1.5, $K_{p,q} \oplus K_{r,s,t}$ is a balanced d-magic graph.

Corollary 5.4 *Let p or q be even positive integers and let r, s and t be even positive integers. If $p = q$ and $r = s = t$, then $K_{p,q} \oplus K_{r,s,t}$ is a supermagic graph.*

Proof. This follows from Theorem 1.1 and Proposition 5.3.

Example 5.1 One considers the tensor product of complete bipartite and tripartite graphs $K_{1,2}$ and $K_{2,2,2}$. A balanced d-magic graph $K_{1,2} \oplus K_{2,2,2}$ is constructed (see Figure 4) and the labels on edges of $K_{1,2} \oplus K_{2,2,2}$ are shown in Table 4

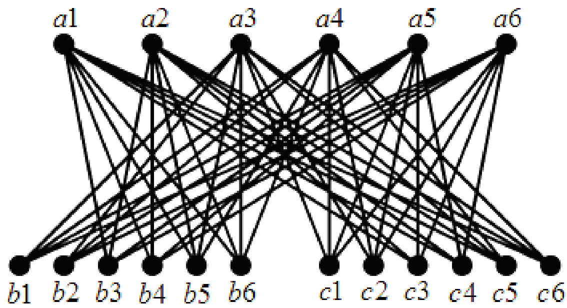

Figure 4 A balanced d-magic graph $K_{1,2} \oplus K_{2,2,2}$.

Table 4 The labels on edges of balanced d-magic graph $K_{1,2} \oplus K_{2,2,2}$.

vertex	b1	b2	b3	b4	b5	b6	c1	c2	c3	c4	c5	c6
a1	-	-	28	22	23	25	-	-	32	18	19	29
a2	-	-	21	27	26	24	-	-	17	31	30	20
a3	36	14	-	-	15	33	40	10	-	-	11	37
a4	13	35	-	-	34	16	9	39	-	-	38	12
a5	44	6	7	41	-	-	48	2	3	45	-	-
a6	5	43	42	8	-	-	1	47	46	4	-	-

Acknowledgements

The author would like to thank the academic reviewer for the careful reading and helpful comments for improving this paper.

References

1. Sedláček J. Theory of graphs and its applications. In: Problem 27. Proc. Symp. Smolenice; Praha; 1963. P.163-164.
2. Stewart BM. Magic graphs. Canad. J. Math. 1966 (18): 1031-1059.
3. Gallian JA. A dynamic survey of graph labeling. Electron. J. Combin. 2009 (16): #DS6.
4. Salehi E. Integer-magic spectra of cycle related graphs. Iranian Journal of Mathematical Sciences and Informatics. 2006 (1): 53-63.
5. Varela MT. On barycentric-magic graphs. Iranian Journal of Mathematical Sciences and Informatics. 2015 (10): 121-129.
6. Bezegová L', Ivančo J. An extension of regular supermagic graphs. Discrete Math. 2010 (310): 3571-3578.
7. Bezegová L', Ivančo J. A characterization of complete tripartite degree-magic graphs. Discuss. Math. Graph Theory. 2012 (32): 243-253.
8. Bezegová L', Ivančo J. On conservative and supermagic graphs. Discrete Math. 2011 (311): 2428-2436.
9. Bezegová L', Ivančo J. Number of edges in degree-magic graphs, Discrete Math. 2013 (313): 1349-1357.
10. Bezegová L'. Balanced degree-magic complements of bipartite graphs. Discrete Math. 2013 (313): 1918-1923.