ปริภูมิเชิงทอพอโลยีแบบเรียบสามัญวางนัยทั่วไป

Generalized Ordinary Smooth Topological Spaces

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บทคัดย่อ

ในบทความนี้ เราได้แนะนำการวางนัยทั่วไปสำหรับปริภูมิเชิงทอพอโลยีแบบเรียบสามัญ ซึ่งเราเรียกว่าปริภูมิเชิงทอพอโลยีแบบเรียบสามัญวางนัยทั่วไป และศึกษาสมบัติบางประการบนปริภูมิเชิงทอพอโลยีแบบเรียบสามัญวางนัยทั่วไป เช่น ตัวดำเนินการปิดคลุม ตัวดำเนินการภายในและความต่อเนื่องของฟังก์ชันบนปริภูมิดังกล่าว

คำสำคัญ: ปริภูมิเชิงทอพอโลยีวางนัยทั่วไป ปริภูมิเชิงทอพอโลยีแบบเรียบสามัญ ปริภูมิเชิงทอพอโลยีแบบเรียบสามัญวางนัยทั่วไป

Abstract

In this paper, we introduce the concept of generalization for ordinary smooth topological space which we call a generalized ordinary smooth topological space and we also study some properties of such space, for instance, closure operator, interior operator and continuity.

Keywords: Generalized topological spaces, Ordinary smooth topological spaces, Generalized ordinary smooth topological spaces.

Introduction and Preliminaries

The concepts of a generalized topology on X was first introduced by Csa´sza´r in as a subset μ of $P(X)$ with the properties[1]:

1. $\emptyset \in \mu$,
2. $\bigcup_{i\in I}\mu_i \in \mu$ for all $\mu_i \in \mu$ and $i \in I \neq \emptyset$.

The pair (X,μ) is called a generalized topological space and μ is called a generalized topology (briefly GT).

In the paper introduced the concepts of ordinary smooth topology on X as a mapping $\tau: 2^X \to I$ with the properties[2]:

$\tau(X) = \tau(\emptyset) = 1$,

$\tau(A \cap B) \geq \tau(A) \wedge \tau(B)$ for all $A, B \in 2^X$,

$\tau(\bigcup_{\alpha\in\Gamma} A_\alpha) \geq \bigwedge_{\alpha\in\Gamma} \tau(A_\alpha)$ for all $\{A_\alpha\} \subseteq 2^X$,

where 2^X is the powerset of X and I is a closed interval [0,1].

The pair (X,τ) is called an ordinary smooth topological space (briefly, $osts$).

In the paper defined an ordinary smooth closure and an ordinary smooth interior in (X,τ) and gave the characterizations of ordinary smooth closure and ordinary smooth interior[2].

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In this paper, we define the space which generalizes the generalized topology on X, we call a generalized ordinary smooth topological space and we also study some properties on such space and continuous maps between the ordinary smooth topological spaces.

Results

In this section, we define a generalized ordinary smooth topological space and give an analogue of generalized ordinary smooth topological space as the result.

Definition 1.1. Let X be a nonempty set. Then a mapping $\mu: 2^X \to I$ is called a generalized ordinary smooth topology (briefly $gost$) on X if μ satisfies the following axioms:

$\mu(\emptyset) = 1$,

$\mu(\bigcup_{\alpha\in\Gamma} A_\alpha) \geq \bigwedge_{\alpha\in\Gamma}\mu(A_\alpha)$ for all $\{A_\alpha\} \subseteq 2^X$,

where 2^X is the powerset of X and I is a closed interval [0,1].

The pair (X,μ) is called a generalized ordinary smooth topological space (briefly $gosts$). We will denote the set of all $gosts$ on X by $GOST(X)$.

Example 1.2. Let $X = \{a,b,c\}$. We define the mapping $\mu: 2^X \to I$ as follows: Let $A \in 2^X$,

$$\mu(A) = \begin{cases} 1, & if\ \ A = \emptyset; \\ 0.8, & if\ \ A = X \text{ or } A = \{b,c\}; \\ 0.6, & \text{if } A = \{a\}; \\ 0.5, & \text{if } A = \{b\} \text{ or } \{a,b\}; \\ 0.4, & \text{if } A = \{c\} \text{ or } \{a,c\}. \end{cases}$$

Then $\mu \in GOST(X)$.

The operators on X which is induced by the generalized ordinary topologies μ are defined as follows:

Definition 1.3. Let (X,μ) be a $gosts$ and let $A \in 2^X$. Then the generalized ordinary smooth closure and generalized ordinary smooth interior of A in X are defined by

$$\overline{A} = \bigcap\{F \in 2^X : A \subseteq F \text{ and } \mu(F^c) > 0\}$$

and

$$A^\circ = \bigcup\{U \in 2^X : U \subseteq A \text{ and } \mu(U) > 0\},$$

respectively.

Example 1.4. From Example 1.2 and let $A = \{a,c\}$.Then

$$A^\circ = \bigcup\{U \in 2^X : U \subseteq \{a,c\} \text{ and } \mu(U) > 0\}$$
$$= \bigcup\{\emptyset, \{a\}, \{c\}, \{a,c\}\}$$
$$= \{a,c\}$$

and

$$\overline{A} = \bigcap\{F \in 2^X : \{a,c\} \subseteq F \text{ and } \mu(F^c) > 0\}$$
$$= \bigcap\{X, \{a,c\}\}$$
$$= \{a,c\}.$$

The following propositions are the properties of $gosts$

Proposition 1.5. Let (X,τ) be a $gosts$ and let $A, B \in 2^X$. Then:

If $A \subseteq B$, then $A^\circ \subseteq B^\circ$ and $\overline{A} \subseteq \overline{B}$.

$(A^\circ)^c = \overline{A^c}$.

$A^\circ = (\overline{A^c})^c$.

$\overline{A} = ((A^\circ)^c)^c$.

$(\overline{A})^c = (A^c)^\circ$.

Proof. (1) Obvious.

(2) For any $A \in 2^X$, we have that $(A^\circ)^c = (\bigcup\{U \in 2^X : U \subseteq A \text{ and } \mu(U) > 0\})^c = \bigcap\{U^c \in 2^X : A^c \subseteq U^c \text{ and } \mu(U^c) > 0\} = \overline{A^c}$

The proof of (3), (4) and (5) are easily obtained from (2).

Proposition 1.6. Let (X,τ) be a $gosts$ and let $A, B \in 2^X$. Then:

$A^\circ \subseteq A$.

$(A^\circ)^\circ = A^\circ$.

$(A \cap B)^\circ \subseteq A^\circ \cap B^\circ$.

Proof. (1) Obvious.

(2) For each $A \in 2^X$, using (1), we have that $(A^\circ)^\circ \subseteq A^\circ$. Since $(A^\circ)^\circ = \bigcup\{U \in 2^X : \mu(U) > 0 \text{ and } U \subseteq A^\circ\} = \bigcup\{U \in 2^X : \mu(U) > 0 \text{ and } U \subseteq \bigcup\{W \in 2^X : \mu(W) > 0 \text{ and } W \subseteq A\}\} \supseteq \bigcup\{U \in 2^X : \mu(U) > 0 \text{ and } U \subseteq A\} = A^\circ$,

then $(A^\circ)^\circ = A^\circ$.

(c) Since $A \cap B \subseteq A$ and $A \cap B \subseteq B$, $(A \cap B)^\circ \subseteq A^\circ$ and $(A \cap B)^\circ \subseteq B^\circ$. Thus $(A \cap B)^\circ \subseteq A^\circ \cap B^\circ$.

Proposition 1.7. Let (X,τ) be a $gosts$ and let $A, B \in 2^X$. Then:

$A \subseteq \overline{A}$.

$\overline{(\overline{A})} = \overline{A}$.

$\overline{A} \cup \overline{B} \subseteq \overline{A \cup B}$.

Proof. The proofs are similar to that of Proposition 1.6.

Definition 1.8. Let (X,μ) be a $gosts$, $r \in I$ and $A \in 2^X$. Then we define $\overline{A_r}$ and A_r° by

$\overline{A_r} = \bigcap\{F \in 2^X : A \subseteq F \text{ and } \mu(F^c) \geq r\}$

and

$A_r^\circ = \bigcup\{U \in 2^X : U \subseteq A \text{ and } \mu(U) \geq r\}$,

respectively.

We called $\overline{A_r}$ a generalized ordinary smooth r-ravel closure and A_r° a generalized ordinary smooth r-ravel interior.

Then the following results are obtained:

Proposition 1.9. Let (X,τ) be a $gosts$ and let $A \in 2^X$. Then:

If $\mu(A) > 0$, then $A = A^\circ$.

If $\mu(A^c) > 0$, then $A = \overline{A}$.

If there is $r \in I_0$ such that $A = \overline{A_r}$, then $A = \overline{A}$.

If there is $r \in I_0$ such that $A = A_r^\circ$, then $A = A^\circ$.

Proof. (1) Let $\mu(A) > 0$. Then

$A \in \{U \in 2^X : U \subseteq A \text{ and } \mu(U) > 0\}$, so

$A \subseteq \bigcup\{U \in 2^X : U \subseteq A \text{ and } \mu(U) > 0\}$,

thus $A \subseteq A^\circ$.

Therefore $A = A^\circ$.

(2) Let $\mu(A^c) > 0$. Then $A^c = (A^c)^\circ$, so $(A^c)^c = ((A^c)^\circ)^c$. Thus $A = \overline{A}$.

(3) Assume that $r \in I_0$ such that $A = \overline{A_r}$. Since $\overline{A} = \bigcap\{F \in 2^X : A \subseteq F \text{ and } \mu(F^c) > 0\} \subseteq \bigcap\{F \in 2^X : A \subseteq F \text{ and } \mu(F^c) \geq r\} = \overline{A_r} = A$, $\overline{A} \subseteq A$. So $A = \overline{A}$.

(4) Assume that $r \in I_0$ such that $A = A_r^\circ$. Since $\mu(A_r^\circ) = \mu(\bigcup\{V \in 2^X : \mu(U) \geq r \text{ and } V \subseteq A\}) \geq \bigwedge \mu(\bigcup\{V \in 2^X : \mu(U) \geq r \text{ and } V \subseteq A\}) \geq r > 0$, $\mu(A_r^\circ) > 0$.

So $A_r^\circ \in \{U \in 2^X : \mu(U) > 0 \text{ and } U \subseteq A_r^\circ\} \subseteq \bigcup\{U \in 2^X : \mu(U) > 0 \text{ and } U \subseteq A\} = A^\circ$.

Thus $A = A_r^\circ \subseteq A^\circ \subseteq A$. Therefore $A = A^\circ$.

2. Generalized ordinary smooth continuity

In this section, we defined a continuous mapping on generalized ordinary smooth topological spaces as follows:

Definition 2.1 Let (X,μ_1) and (Y,μ_2) be $gosts's$. Then a mapping $f: X \to Y$ is said to be:

A generalized ordinary smooth continuous (briefly $gos-continuous$) if $\mu_2(A) \leq \mu_1(f^{-1}(A))$ for all $A \in 2^Y$.

A generalized ordinary weakly smooth continuous (briefly $gows-continuous$) if for each $A \in 2^Y$, $\mu_2(A) > 0 \Rightarrow \mu_1(f^{-1}(A)) > 0$.

Example 2.2. Let $X = \{a,b,c\}$. We define two mapping as follows: For each $C, D \in 2^X$,

$$\mu_1(C) = \begin{cases} 1, & if\ C = \emptyset; \\ \frac{1}{2}, & if\ C = X \text{ or } C = \{b,c\} \text{ or } C = \{a\}; \\ 0, & otherwise, \end{cases}$$

and

$$\mu_2(D) = \begin{cases} 1, & if\ D = \emptyset; \\ \frac{1}{3}, & if\ D = X \text{ or } D = \{b,c\} \text{ or } D = \{a\}; \\ 0, & otherwise. \end{cases}$$

Cleary, the identity mapping $id: (X,\mu_2) \to (X,\mu_1)$ is $gows-continuous$, but id is not $gos-continuous$.

The following results are obtained that:

Corollary 2.3. Let (X,μ_1) and (Y,μ_2) be $gosts's$ and let a mapping $f: X \to Y$. Then: f is $gos-continuous$ iff $\mu_2(A^c) \leq \mu_1(f^{-1}(A^c))$ for all $A \in 2^Y$. f is $gows-continuous$ iff $\mu_2(A^c) > 0 \Rightarrow \mu_1(f^{-1}(A^c)) > 0$ for all $A \in 2^Y$.

Proposition 2.4. Let (X,μ_1) and (Y,μ_2) be $gosts's$ and let a mapping $f: X \to Y$ be $gows-continuous$. Then:

$f(\overline{A}) \subseteq \overline{f(A)}$ for all $A \in 2^X$.

$\overline{f^{-1}(B)} \subseteq f^{-1}(\overline{B})$ for all $B \in 2^Y$.

$f^{-1}(B^\circ) \subseteq (f^{-1}(B))^\circ$ for all $B \in 2^Y$.

Proof. (1) Let $A \in 2^X$. Since

$f^{-1}\left(\overline{f(A)}\right) = f^{-1}(\bigcap\{F \in 2^Y : \mu_2(F^c) > 0 \text{ and } f(A) \subseteq F\})$

$= \bigcap\{f^{-1}(F) \in 2^X : F \in 2^Y, \mu_2(F^c) > 0 \text{ and } A \subseteq f^{-1}(F)\}$

$\supseteq \bigcap\{f^{-1}(F) \in 2^X : F \in 2^Y, \mu_1(f^{-1}(F^c)) > 0 \text{ and } A \subseteq f^{-1}(F)\}$

$= \overline{A}$,

then $\overline{A} \subseteq f^{-1}\left(\overline{f(A)}\right)$.

Thus $f(\overline{A}) \subseteq f\left(f^{-1}\left(\overline{f(A)}\right)\right) \subseteq \overline{f(A)}$.

(2) Let $B \in 2^Y$, we have $f^{-1}(B) \in 2^X$.

Then $f\left(\overline{f^{-1}(B)}\right) \subseteq \overline{f(f^{-1}(B))} \subseteq \overline{B}$,

so $\left(\overline{f^{-1}(B)}\right) \subseteq f^{-1}\left(f\left(\overline{f^{-1}(B)}\right)\right) \subseteq f^{-1}(\overline{B})$.

(3) Let $B \in 2^Y$.

Then

$f^{-1}(\overline{B^c}) = f^{-1}((B^\circ)^c) = (f^{-1}(B^\circ))^c = \left(f^{-1}\left(\left(\overline{B^c}\right)^c\right)\right)^c = f^{-1}(\overline{B^c}) \supseteq \overline{f^{-1}(B^c)} = \overline{(f^{-1}(B))^c} = ((f^{-1}(B))^\circ)^c$

.

So $\left(\left(f^{-1}(B)\right)^{\circ}\right)^{c} \subseteq f^{-1}((B^{\circ})^{c})$.
Hence $f^{-1}(B^{\circ}) \subseteq \left(f^{-1}(B)\right)^{\circ}$.

The following Corollary is immediate from Definition 2.1 and Proposition 2.4.

Corollary 2.5. Let (X,μ_1) and (Y,μ_2) be $gosts's$ and let a mapping $f: X \to Y$ be $gos-continuous$. Then:

$f(\overline{A}) \subseteq \overline{f(A)}$ for all $A \in 2^X$.
$\overline{f^{-1}(B)} \subseteq f^{-1}(\overline{B})$ for all $B \in 2^Y$.
$f^{-1}(B^{\circ}) \subseteq \left(f^{-1}(B)\right)^{\circ}$ for all $B \in 2^Y$.

The generalized ordinary smooth open map and generalized ordinary smooth closed map are defined as follows:

Definition 2.6. Let (X,μ_1) and (Y,μ_2) be $gosts's$. Then a mapping $f: X \to Y$ is said to be: a generalized ordinary smooth open (briefly $gos-open$) if $\mu_1(A) \le \mu_2(f(A))$ for all $A \in 2^X$. a generalized ordinary smooth closed (briefly $gos-closed$) if $\mu_1(A^c) \le \mu_2(f(A^c))$ for all $A \in 2^X$.

Example 2.7. Let $X = \{a,b,c\}$. We define two mapping as follows: For each $C, D \in 2^X$,

$$\mu_1(C) = \begin{cases} 1, & if\ C = \emptyset; \\ \frac{1}{4}, & if\ C = X; \\ \frac{1}{6}, & if\ C = \{b,c\}; \\ 0, & otherwise, \end{cases}$$

and

$$\mu_2(D) = \begin{cases} 1, & if\ D = \emptyset; \\ \frac{1}{2}, & if\ D = X; \\ \frac{1}{5}, & if\ D = \{b,c\}; \\ 0, & otherwise. \end{cases}$$

Then $\mu_1, \mu_2 \in GOST(X)$. Consider the identity mapping $id: (X,\mu_1) \to (X,\mu_2)$. Then we can see that id is $gos-open$ and $gos-closed$.

Then we obtain the following result:

Proposition 2.8. Let (X,μ_1) and (Y,μ_2) be $gosts's$. If $f: X \to Y$ is $gos-open$, then $f(A^{\circ}) \subseteq \left(f(A)\right)^{\circ}$ for each $A \in 2^X$.

Proof. Let $A \in 2^X$. Since
$f(A^{\circ}) = f(\cup\{U \in 2^X : \mu_1(U) > 0 \text{ and } U \subseteq A\})$
$= \cup\{f(U) \in 2^Y : U \in 2^X, \mu_1(U) > 0 \text{ and } f(U) \subseteq f(A)\}$
$\subseteq \cup\{f(U) \in 2^Y : U \in 2^X, \mu_2(f(U)) > 0 \text{ and } f(U) \subseteq f(A)\}$
$\subseteq \cup\{V \in 2^Y : \mu_2(V) > 0 \text{ and } V \subseteq f(A)\}$
$= \left(f(A)\right)^{\circ}$,
$f(A^{\circ}) \subseteq \left(f(A)\right)^{\circ}$.

Definition 2.9. Let (X,μ_1) and (Y,μ_2) be $gosts's$. Then a mapping $f: X \to Y$ is called a generalized ordinary smooth homeomorphism if f is a bijective and f, f^{-1} are generalized ordinary smooth continuous.

Now, we have the relation of generalized ordinary smooth homeomorphisms, $gos-open$ and $gos-closed$ as follow:

Theorem 2.10. Let (X,μ_1) and (Y,μ_2) be $gosts's$ and let $f: X \to Y$ be a bijective and f be $gos-continuous$. Then the following statements are equivalent:

f is generalized ordinary smooth homeomorphism.
f is $gos-open$.
f is $gos-closed$.

Proof. (1)⟹(2) Assume that f is a generalized ordinary smooth homeomorphism. Then $\mu_1(A) \le \mu_2((f^{-1})^{-1}(A)) = \mu_2(f(A))$. Thus f is $gos-open$.

(2)⟹(3) Assume that f is $gos-open$. Let $A \in 2^X$, we have $\mu_1(A^c) \le \mu_2(f(A^c))$. Since f is bijective, $\mu_1(A^c) \le \mu_2(f(A^c))$. Thus f is $gos-closed$.

(3)⟹(1) Assume that f is $gos-closed$. Let $A \in 2^X$. Then $\mu_1(A) \le \mu_2(f(A)) = \mu_2((f^{-1})^{-1}(A))$. Thus f^{-1} is $gos-continuous$. Hence f is a generalized ordinary smooth homeomorphism.

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